

INTRODUCTION TO OPTIMIZATION

The words *optimize* and *optimization* are relatively new to the English language.

optimize *tr.v.* 1. To make as perfect or effective as possible. 2. To make the most of. (*American Heritage Dictionary of the English Language*, Third Edition, 1992.)

optimization *n.* The procedure or procedures used to make a system or design as effective or functional as possible, especially the mathematical techniques involved. (*American Heritage*, Third Edition, 1992.)

What other words do you know that begin with “optim,” and how are their meanings related?

People create mathematics to help find the best solutions to practical as well as theoretical problems. It is no wonder, then, that the theme of optimization shows up repeatedly in mathematics. Much mathematics has been developed in attempts to answer optimization questions like these:

- What’s the fastest way?
- What’s the shortest way?
- What’s the smallest that. . . ?
- What’s fairest?
- What’s the cheapest way?
- What’s the longest way?
- What’s the biggest that. . . ?

For example:

- Where should a straight highway be built to pass as close as possible to five different towns?
- How much should a cookie company charge for each box of cookies if it wants to maximize its profit?
- The most expensive part of framing a house is constructing the walls. How can I build a house with the most floor space if I can afford a perimeter of at most 128 feet? (How can I maximize floor space for a given cost?)
- What’s the shortest route for driving from New York to Chicago?

Why wouldn’t it just charge \$100, or \$1000, per box?

There is an interesting article about this problem in the journal *Mathematics Teacher*. (Cunningham, Robert F., "What manufacturers say about a max/min problem," *Mathematics Teacher* 87 (no. 3) 1994, pp. 172–175.)

A pound is 16 ounces.

Maximize what you pack into the jars and minimize the amount you have left over.

£ is the symbol for British pounds.

Does it matter how much money you *have*? What if you have £10, or £300?

A beautiful book, *Maxima and Minima without Calculus* by Ivan Niven (Dolciani Mathematical Expositions No. 6; Washington, D.C.: MAA, 1981), has an approach to optimization that uses algebra.

- How can a manufacturer of containers maximize the volume of a container while minimizing the amount of material needed to make the container?

FOR DISCUSSION

What makes these problems “optimization” problems? Think of three other optimization problems.

Here are some optimization problems to get you started.

1. A food pantry receives a donation of 17 pounds of peanut butter from the Good and Nutty company, all in one big tub. It needs to pack the peanut butter into two sizes of jars. The big one holds two pounds and the small one holds 9.5 ounces. How many jars of each should be packed so that the minimum amount of peanut butter is left over?
2. A visiting friend from London needs to change British pounds to American dollars. A currency exchange service will give you \$1.55 for each £1 with no service charge. The bank down the street will give you \$1.65 for each £1, but charges \$2 for the transaction. Where should you go to change the money?

One of the most useful mathematical tools for solving optimization problems is *calculus*. In fact, much of calculus was invented to solve optimization problems.

However, many optimization problems can be solved without calculus. Sometimes algebra, arithmetic, and a little experimentation is enough. In this module, you will develop *geometric* and *visual* techniques for solving some optimization problems.

MAKING THE LEAST OF A SITUATION

Optimization often involves *minimizing* something—making it as small as possible. One might minimize rate, time, or the number of ways something can happen. In geometry, the things to minimize tend to be measures like distance, length, area, volume, or angle measure. Minimization problems occur in many fields such as architecture, computer programming, economics, engineering, and medicine.

FOR DISCUSSION

The *distance* between two fixed points is the length of the shortest path between them. It cannot be minimized or maximized; it just is what it is. A *path* between the two points can be as long as one wants, but can't be shorter than the distance between those two points.

What is the difference between the terms *distance* and *length*?

Humble as the subject seems, there's a lot to think about here. For example, describe the shortest path between two points on a flat surface, or plane. How many distinct paths have that shortest length? How long might the longest path be? How might such a path look? How many of these are there?

Now think about two points on a ball. Describe the shortest path on the surface of the ball between the north pole and south pole. How many distinct paths are that short? What about a shortest path between the north pole and some point on the equator? Or what about such a path between two points on a cylinder?

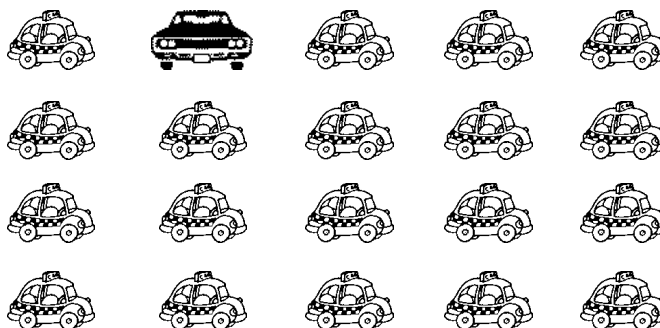
Write up the ideas that you generated in your discussion.

Finding the shortest path between two points might seem to be the simplest minimization problem in geometry, but sometimes you can't take the shortest path because things are in the way.

Is there more than one best route to the car?

- Suppose you and a friend are at a corner of a parking lot and it's raining fiercely. All the spaces are taken, and you want to get to your friend's car the shortest way possible. You can walk or run around the cars or between them. What route minimizes the length of your path to the car?

Your car



You

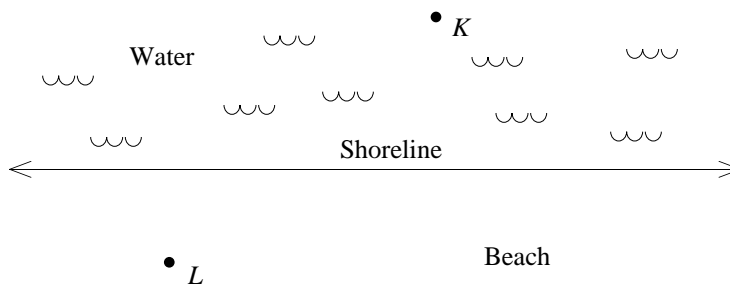
Does it matter which way the cars are facing?

How do you get to the car quickly?

MINIMIZING DISTANCE

These problems help you develop some minimization strategies.

- You are lounging on the beach at L , and you want to run to the shoreline and swim out to your friends on a raft at K .



This problem will come up again. We'll call it the run-and-swim problem.

A run and a swim

Make sketches to show where you should hit the shoreline.

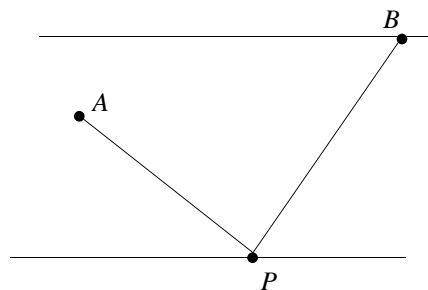
- You want the swim to be as short as possible. To what point on the shoreline should you run to minimize the path you swim? Why?
- You want to reach K with the least running. At what point on the shoreline should you enter the water now? Why?
- You want to reach K in the least possible total distance. *Now* where should you hit the shoreline? Why?

In the parking lot problem, cars were in the way.

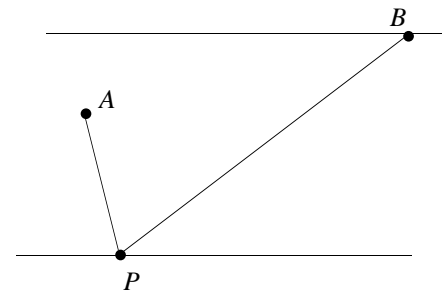
Sometimes you can't take the shortest path because things are in the way; sometimes you don't take the shortest path because you care about the means by which you get yourself there. Still other times you don't take the shortest path from one point to another because there is a third place you want to visit first.

Imagine, for example, that you're motorboating on a river, and the boat is very short on fuel. You *must* drop a passenger off on one riverbank first, and then you can go to refuel at a station on the other riverbank.

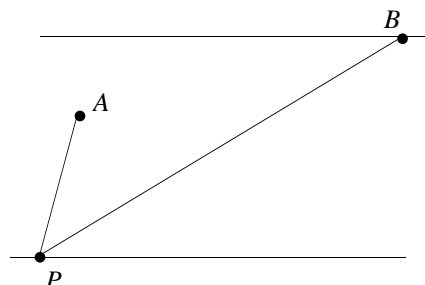
Here's a picture of the situation. The boat is at A . You will drop off the passenger on the lower bank (we'll call that not-yet-chosen place P), and then you will refuel on the opposite bank at B . *You* choose the location for P (you can drop the passenger off anywhere on that bank), but of course, you want to minimize total distance because you're low on fuel.



Should you land here?



... or here?



No ... definitely not there.

... or maybe here?

Be sure that you keep records as you work, and write down your conjectures even if you later reject them. You might look at what happens to AP and PB separately.

- Study what happens to the sum $AP + PB$ as P moves along the riverbank, and find the optimal place for P by experimentation. Also, find a way to locate the best spot for P that *doesn't* involve trial and error. Explain and justify your solutions.

..... **WAYS TO THINK ABOUT IT**

Imagine the sum of the two distances as a system that involves two points A and B , that are fixed, and one point, P , whose position can move along a line. (B is also pictured as on a line, but that's irrelevant. The refueling station's position doesn't move; it might as easily be on a tiny island as on the riverbank.)

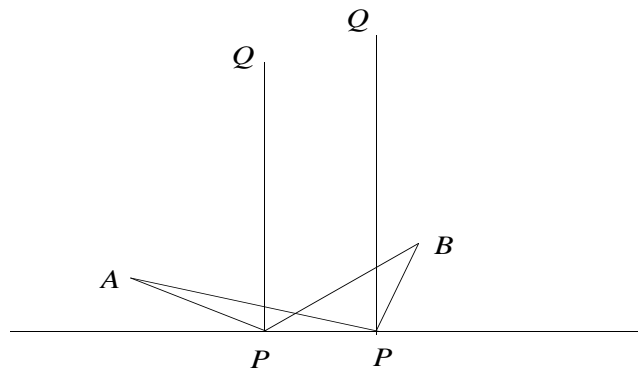
In this system, several things are changing. As P moves, the lengths AP and PB change (while P is “in between” A and B , one length grows as the other shrinks). The sum $AP + PB$ also changes. The relationship connecting every position of P to its associated sum $AP + PB$ is an example of a *function*.

The previous pictures show the “setup” of the system. There are (many) other ways of picturing the system's behavior. Problem 4 shows one way.

.....

This particular function can be thought of as a little machine that rides with P along the riverbank, always calculating $AP + PB$.

4. On graph paper, draw a horizontal line to represent the lower riverbank. Place points A and B above it to represent the original locations of the boat and the refueling station. Now, for each P that you select along the riverbank, plot a point Q directly above it so that the distance from Q to P is the same as the sum $AP + PB$. The picture that follows shows two possible positions of P and the corresponding positions of Q .



Geometry software can display the *numerical* value of the sum of the distances as well.

After you have plotted enough of these positions for Q , you can sketch a curve along them to suggest the behavior of the system. As P moves along the riverbank from, say, directly below A to directly below B , what happens to the height of Q above P (the sum of the distances)?

You can use geometry software to make the same graph. This has the advantage of being a *dynagraph* (a dynamic graph): it allows you to drag P back and forth to generate the graph and to get a sense of the continuous behavior of the system.

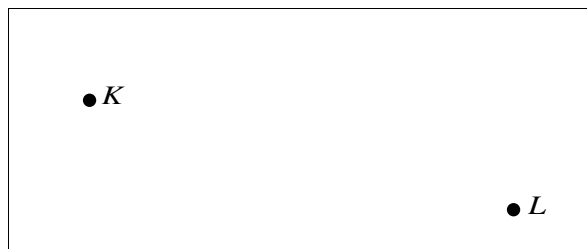
5. How do the positions of A and B , relative to the lower bank, affect your answer to Problem 3? For example, if A and B are exactly the same distance away from the lower riverbank, where is the best location for P ? If A is half as far as B is from the lower bank, where is the best location for P ?
6. If you hold A constant, but widen the river to move B farther away, does the best location for P move nearer to A or farther from A ?

Does your method work for any polygonal pool?

What does *polygonal* mean?

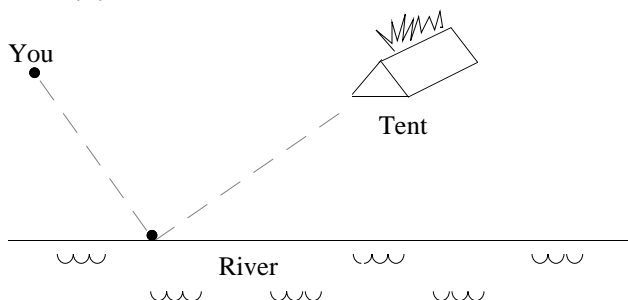
We'll refer to this problem again. Let's call it the "burning tent" problem.

7. You're in a rectangular swimming pool at K , out of reach of the sides of the pool. Before swimming to L , you want to swim to a side of the pool to put down your sunglasses. Explain how to find the place to put your sunglasses that minimizes the length of the path you swim.

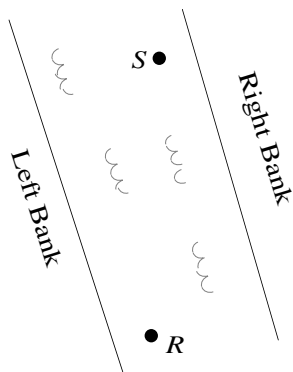


Don't lose your sunglasses.

8. Returning from a hike while on a camping trip, you see that your tent is on fire. Luckily, you're holding a bucket and you're near a river. Where along the river should you get the water to minimize the total distance you travel to get back to the tent? Justify your answer.



Where should you stop to fill the bucket?



Back and forth ...

9. In the picture at the side, a canoe is in a river at R . It must first let off a passenger on the left bank, then pick up a passenger who will meet the canoe on the right bank, and finally deliver that passenger to an island at S . Explain how to find the drop-off and pickup points that minimize the total distance traveled. Check it using a compass and ruler, string, or geometry software.

CHECKPOINT.....

10. Define each of these words:

optimization

minimum

maximum

perpendicular

bisect

polygon

distance

length

path

11. What is the shortest way to get from one point to another?

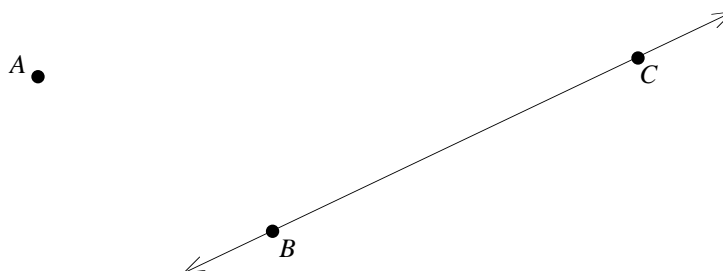
12. What is the shortest way to get from a point to a line?

13. How would you find the shortest path to get from A to:

a. the line \overleftrightarrow{BC} ?

b. the segment \overline{BC} ?

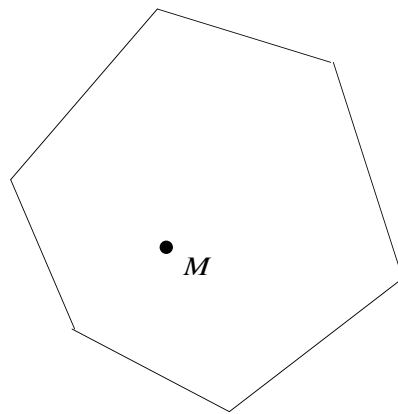
Explain your reasoning.



14. Describe your general *strategy* for minimizing the total path from a point to a line and then to another point not on that line.
15. Write what you know about perpendicular bisectors and describe how you can use them to help you solve an optimization problem.

TAKE IT FURTHER.....

16. You are at an arbitrary point M in a strange swimming pool with many sides. How would you find the shortest way out of the pool? Would you go to a corner or to a side?

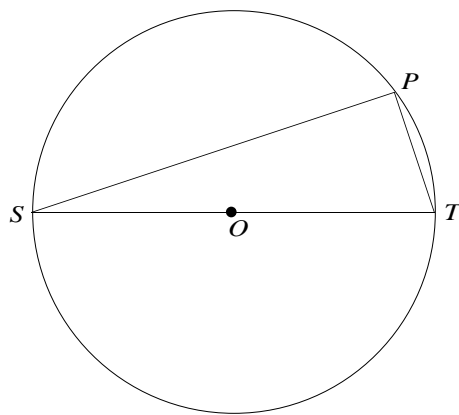


How would you describe all the points that are a *given* distance away your spot in the pool?

Why would this be an important problem for navigators?

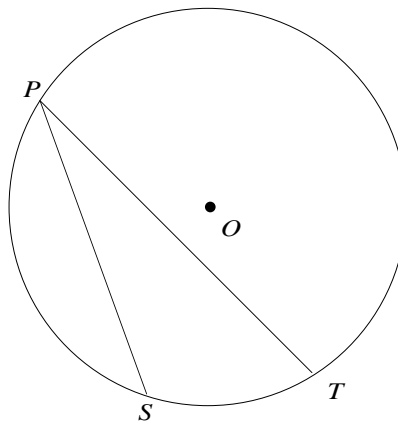
17. Can you draw a pool or a position for M in such a way that a person would choose to swim to a corner as the shortest way out?
18. Mark two points on a basketball or some other sphere. What's the shortest path from one point to the other if you have to stay on the surface of the sphere?

19. Points S and T are the endpoints of a diameter of the circle O . What are the possible values for $m\angle SPT$? Where should you put P on the circle to minimize $SP \times PT$? To maximize $SP \times PT$?

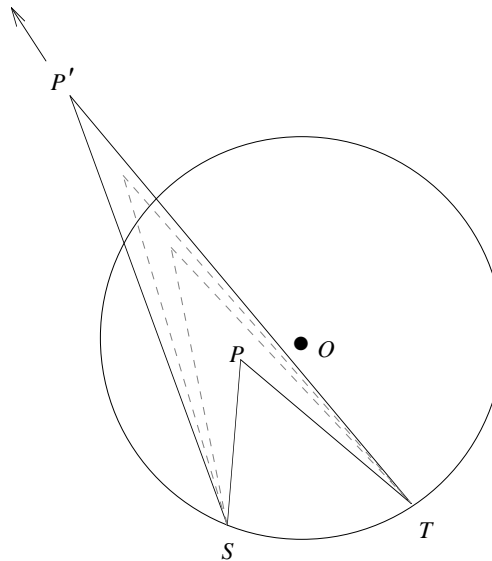


TAKE IT FURTHER.....

20. Assume that points S and T are fixed in the positions shown below, and that P is free to roam around the circle. ($\angle SPT$ is called an *inscribed angle* in the circle.) What are the possible values for $m\angle SPT$? Where should you put P to minimize $SP \times PT$? To maximize $SP \times PT$?



21. Again, points S and T are fixed, but this time P is free to move off the circle. Suppose that P starts inside the circle and gradually moves to be outside the circle. What happens to the size of $\angle SPT$ in the process? Why?



THEOREM 6.1 *The Triangle Inequality*

The sum of the lengths of any two sides of a triangle is always greater than the length of the third side.

22. What does this theorem have to do with Problem 2 (the run and swim problem) or with any of the other problems in this investigation? Restate the theorem in your own words. Justify your statements.
23. Is the statement below true or false? Justify your answer.

In any quadrilateral, the sum of the lengths of the diagonals is smaller than the perimeter of the quadrilateral and bigger than half its perimeter.

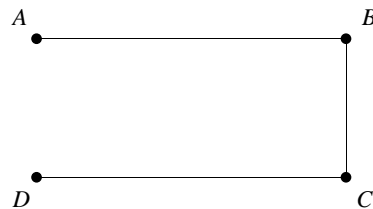
24. A hungry spider sees a bug on the ceiling. The picture below shows the spider on the left wall of the room. What would be the shortest path along the walls between the spider and the bug? (No air travel is allowed.) Make a sketch to show what you think is the shortest path, but make sure to *describe how* to find the shortest path as well.



Which way?

A **network** is a diagram consisting of a set of points and the segments or arcs connecting them.

25. Electricians, computer scientists, and others look for optimal ways to connect various points to make a network.
- Using the rectangular set of points below, can you create a network whose connecting horizontal and vertical paths have a total length less than the one shown?
 - What if the restriction to horizontal and vertical segments is removed? Find another network that uses segments that are not necessarily horizontal or vertical and that has a total length less than the one shown.



How many paths are there if it matters where you start (so that $ABCD$ is different from $DCBA$)?

PERSPECTIVE ON STUDENT RESEARCH EXPERIENCE

Can college students do original research in mathematics? What kinds of career opportunities are available for math majors? In this essay, you will read about one student's experiences and outlook.

Problems 24 and 25 are related to some of the questions Tonya McLean investigated at a summer mathematics research project for undergraduates. Tonya sent us those two problems, along with the two problems at the end of the following essay, as an introduction to the minimal networks topic that she and her colleagues researched. The students worked on a lot of different problems to gain insight and experience into their main challenge: finding algorithms that would help them create truly minimal solutions after getting *approximate* solutions (say, from a computer) for a minimal network of connected points. A solution might involve minimizing distances, lengths, or cost. Below is Tonya's description of her experiences in finding and participating in the research project.

"Throughout my first two years in college, I only rarely thought about how I would use my education as a math major. My main choices were entering the job market or continuing my studies with graduate school. I knew I might have a lot of job opportunities with my undergraduate degree, even if I wasn't sure what they might be, yet somewhere in the back of my mind sat the question of graduate school. As my junior year progressed, however, I found myself continually trying to decide whether I should even consider going to graduate school—I wasn't sure it would be interesting or rewarding. No matter how much I thought about it, nothing ever seemed to convince me of a decision. Luckily, I soon stumbled upon an opportunity to do summer research at the undergraduate level; that was just what I needed to complete my decision-making process.

"During each of the three summers prior to my junior year, I had worked at the grocery store in my hometown. Now I was determined to find a better job, a job where I could use my mind and finally get some reward for my three years of higher education. What can a math major possibly do for summer employment? Plenty! Sadly though, I did not know this yet. I didn't even know where to begin looking for a new summer job.

"After asking around, I learned that the campus Career Development Office had a binder with listings of summer opportunities available for math majors. I found listings of colleges which had programs offering undergraduates an experience in conducting research in mathematics. Each of the abstracts which described the research to be conducted was intriguing, and most could be understood without looking up any of the terms in a textbook. How perfect! Since conducting research is a key ingredient in attending graduate school, not only would this experience provide me

with a challenging and rewarding summer, it would also help me decide what path to pursue after college.

“I applied to several of the research programs, and was accepted at one called SMALL, located at a school in Massachusetts. I was assigned to work with two research teams. Each team consisted of five undergraduates and a faculty advisor. One undergraduate for each team was selected to become the group leader, whose responsibility was to facilitate communication among group members and ensure that the group’s research was moving along in the desired direction. Each group met daily to discuss any progress that individual members had made since the last meeting, and to determine what the next step would be to bring the group closer to a solution. When we weren’t meeting as a group, we worked alone on the problems, but if at any point a member needed help, he or she would consult other members of the group for guidance.

“I was a member of both the Minimal Networks Group and the Symmetry Group, and I was also the group leader for the Symmetry Group. Alice Underwood was the faculty advisor for the Minimal Networks Group. We spent the first week of the program working on a few introductory problems and deciding on a general direction for the group. The central problem we addressed was finding a formula that would provide the minimal solution of connecting a network, given an approximate solution for that network. At the end of the program, it was exhilarating to have actually solved a problem which we had seen for the first time only weeks earlier.

“Robert Mizner was the faculty advisor for the Symmetry Group. The project for this group was built around the book by B. Sagan *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*. This was a graduate level algebra text that we were reading. We read and discussed the text extensively, and compiled a paper that was designed to be an introduction to symmetric groups—a guide for students unfamiliar with the topic. We also added solutions to relevant problems in order to aid the reader in solidifying concepts.

“Prior to participating in SMALL, if anyone had asked me what mathematics research was like, I would have been speechless. Similarly today, when discussing what I do, I often encounter people who are dumbfounded when they learn that there are academics and business professionals who are currently conducting new research in mathematics and making new discoveries. Some feel that mathematics is a lost subject that doesn’t help us with anything but balancing our checkbooks. Nothing could be more wrong. Mathematics is just as important as other sciences, like chemistry and physics, and can be applied to solve most any problem.

Many parts of algebra are concerned with the ways various objects can be combined. In *symmetric groups* the “objects” are the ways to rearrange things—these are called *permutations*. The *results* of the possible permutations on the symbols a, b, c are $\{abc, acb, bac, bca, cba, cab\}$. See the example at the very end of this investigation if you want to know a little more about symmetric groups.

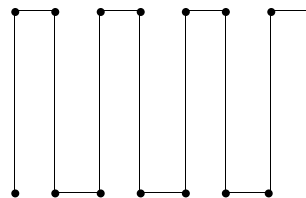
The “SMALL”

Undergraduate Research Project (named after participating professors) is probably the largest in the country. In this summer program, undergraduates spend nine or ten weeks working on problems of current research interest, proving theorems, and writing up results, often for publication and presentation.

Approximately seven or eight groups of five or six students each, work in areas such as geometry, dynamical systems, knot theory, group theory, and quantum mechanics (a new physics component). Funding from the National Science Foundation’s REU program (Research Experiences for Undergraduates), NECUSE (New England Consortium for Undergraduate Science Education), individual colleges, universities, and other sources provides some money for student stipends and expenses. Contact the National Science Foundation for a list of REU sites if you think you might want to apply.

“My experience at SMALL helped me decide to attend graduate school. I am currently enrolled in a Master of Arts program. As a teaching fellow at my college, I have encountered undergraduates who tell me they enjoy their mathematics courses, yet they don’t choose to major in mathematics because they “can’t do anything but teach” with a math major. It is wonderful to see some relief in their faces when I tell them that mathematics majors literally can do anything and enter into any fields; that the academic and professional worlds deeply respect the kind of training, curiosity, and mental discipline that mathematics majors develop. (Law schools, for example, actively recruit mathematics majors because of their training and the ways of thinking they have developed.) I urge those who enjoy mathematics to speak to mathematics professors, other professionals, and/or a career counselor or two before deciding on a college or a job path, for I have found that mathematics does nothing but open doors for one’s career.”

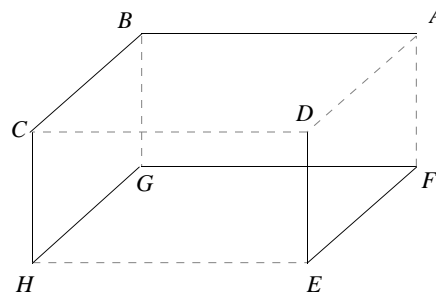
- 26.** Assume that only horizontal and vertical paths may be used to connect the set of points contained in the network below. Can you come up with a *better* solution (a shorter path) than the one depicted? Can you find a minimal solution? If so, be sure to explain how you know it is minimal.



Connected points

- 27.** Information may be transmitted only along the edges of the box shown on the next page. The box has a square base, and a height that’s half the length of either of the other two dimensions. The manufacturer of this device wants to connect all of the vertices. Information sent along the path used should “visit” each vertex once and only once. The solution shown here in solid lines could be written as $ABCHGFED$.
- Find a better solution (one that has a shorter total path) than the one shown.
 - What would be a worse solution (one that has a longer total path than the one shown)?
 - Find a best (minimal) solution and explain why it is best.

- d. If the connection between the vertices doesn't have to follow a continuous path from one vertex to the next (without retracing some part of the path), is there a solution that is shorter in total length than the one you found in part c? Explain.
- e. How do your answers change if the box has a nonsquare rectangular base?



Here, information can travel in any direction along the edges.

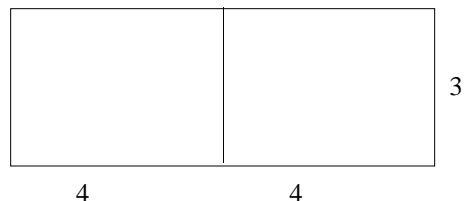
In another “Student Research Experience” section in Investigation 6.21, you will find an essay by Scott Greenleaf, who participated in the same summer research program as Tonya McLean. One of the problems faced by Scott’s research team was to transform shapes in a way that reduced the total perimeter without reducing the area. The problem below illustrates a typical situation they came up against. In this one, the team was actually concerned with *two* adjacent regions, the shape and position of which could change, but not the area.

28. Suppose that a farmer has two kinds of animals and keeps them in two connected rectangular pens. The farmer wants to rebuild the pens according to the following constraints:
- The shape of each pen can change, but the pen must retain the same area;
 - The sides of the pens must meet at right angles.

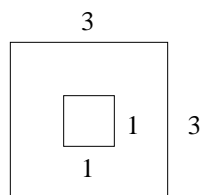
The shapes and measurements of the original pens are shown on the next page. How should each pair of pens be rebuilt to minimize the amount of fencing

needed? Draw a picture of the new pens that minimizes total perimeter and indicate the sidelengths. Explain in your own words how, or where, the new combination of shapes reduces the perimeter.

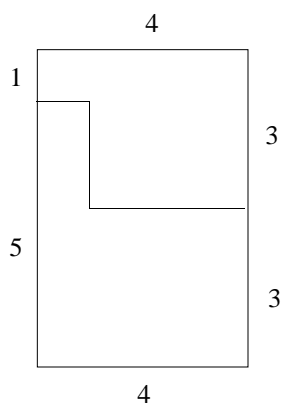
a.



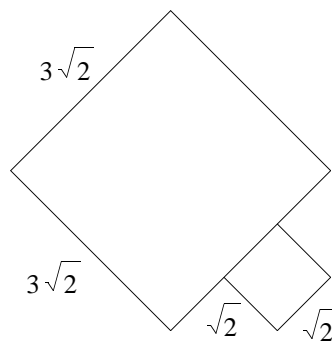
b.



c.



d.



In this problem, the shapes model adjacent pens. When the perimeters are minimized with regard to area, however, the solutions also model the shapes in which certain types of double crystals grow.

WHAT IS A SYMMETRIC GROUP?

Mathematics is often concerned with finding a common underlying structure in things that may appear quite different. For example, consider the four arithmetical operations on integers: addition, subtraction, multiplication, and division. At one level they're all "the same"; they're arithmetic operations. At another level, they're all different. There are a couple of useful ways to classify them as pairs, but there's something that applies to exactly *three* of them that is a particularly important mathematical idea. That important property is called *closure*. For integers, closure means that applying the operation to two integers always results in another integer. The set of integers is "closed" with respect to three of the operations because you don't have to look outside the set for the *result* whenever you apply one of them. Which three operations do you think they are?

When a set (like the integers) with one operation (like multiplication) has this closure property and a few other properties, mathematicians call it a *group*. The *symmetric groups* are an important class of groups that involve *permutations*—rules (or functions) that arrange and rearrange some objects. These groups arise naturally in algebra as well as in geometry, in crystallography, and in physics.

There are many ways to shuffle a set of cards. For example, when you have, say, three cards, every shuffle produces one of six possible arrangements of the cards. (What are they?) Two shuffles can be "composed" by doing the first one and then doing the second one. The mathematical system whose elements are the card shuffles, and whose operation is this kind of composition, is an example of the *symmetric group on three symbols*. How many distinct shuffles are there on four cards? (Two shuffles are considered the same if they produce the same arrangement of cards.) How many distinct shuffles are there on five cards? On n cards?

MAKING THE MOST OF A SITUATION

Just as you sometimes want to minimize things, there are times you want to *maximize* things—to make things as large as possible. In general, people want to minimize things that cost money, require a lot of boring work, waste time, or make them uncomfortable. People often want to *maximize* things that make food or money, involve a lot of interesting work, save time, or make them more comfortable.

FOR DISCUSSION

Perhaps the most common thing geometers try to maximize is area. Describe some situations where maximizing area would help in making food or money, saving time, or providing more interesting work or more comfort.

The most basic area-maximization problem is this:

For a historical discussion of this problem, see Investigation 6.20.

For all shapes with the same perimeter, which one has the greatest area?

Many of the problems in this investigation can help you answer this question. For now, make some conjectures and give explanations for your conjectures. Don't restrict your ideas only to polygons, because "shapes" means figures formed by curves as well.

SOME MAXIMIZATION PROBLEMS

These problems will help you develop some maximization strategies.

1. Triangles of many different sizes may have two sides of lengths 5 and 6. Which of these triangles encloses the most area?
2. Of all the parallelograms whose sidelengths are 20, 30, 20, and 30, which one encloses the most area?

3. Suppose you want to build a house with some sort of rectangular base. The most expensive part of framing the house is the walls. Given your budget, you decide you can afford a house whose base (or floor) has a total perimeter of 128 feet. What dimensions should you choose for the floor of the house if you want to maximize the floor area?
4. A friend came up with the following argument which shows that a square with sidelength 32 feet is the best solution to Problem 3:

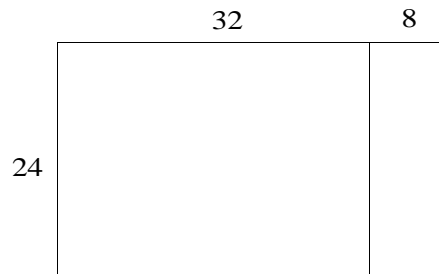
This is one of those problems where you have to read a proof and understand it, so read carefully.

Is this a good example?
(What's the perimeter of a 40×24 rectangle?)

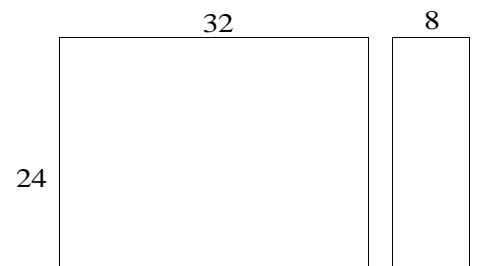
I'll show that a 32×32 square is best by demonstrating that any other rectangle with a perimeter of 128 has a smaller area than the square. I'll do this by showing that I can cut up such a rectangle and make it fit inside the 32×32 square with room to spare.

Suppose, for example, I have a 40×24 rectangle.

First, I'd cut it like this:

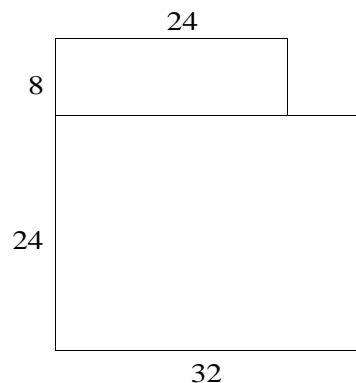


Snip!

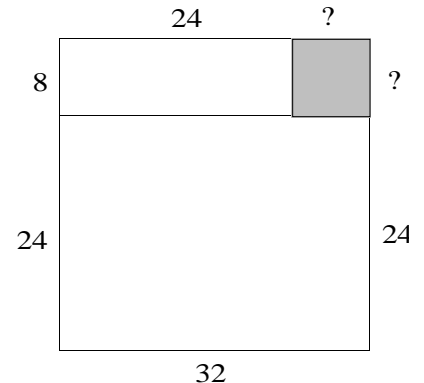


Then rearrange these ...

Then, I take the small strip off the side, turn it, and put it on top of the 24×32 rectangle:



... to get this ...



... which is not quite a square.

See the *Connected Geometry* module *The Cutting Edge* for more examples of this technique.

See? The operation *rearranged* the area, but I didn't add or lose any area. Meanwhile, my two pieces cover *some* of the 32×32 square, but not all of it. The shaded part isn't covered, so the square has more area! Therefore, the area of the 32×32 rectangle is *bigger* than the area of the 40×24 rectangle.

This proof works for the 40×24 rectangle. Critique the proof. Does it generalize to *all* rectangles of perimeter 128? The suggestions listed below may help. Use them only if they *do* help.

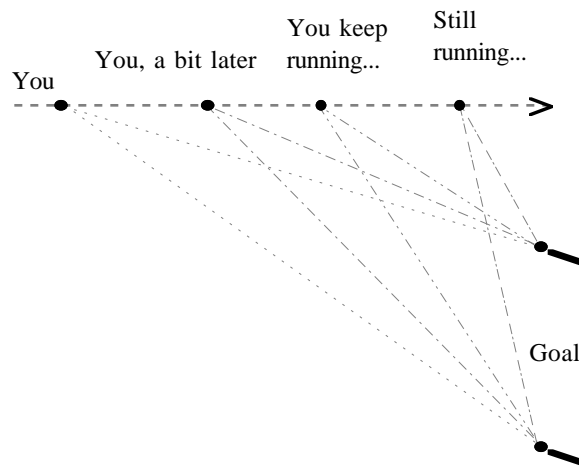
- a. Does the cutting argument for the 40×24 rectangle really work? Rewrite it in your own words; explain each step.
 - b. Try using a similar argument for a rectangle with different dimensions but with the same perimeter of 128 feet.
 - c. Use the cutting argument to show that *any* nonsquare rectangle with a perimeter of 128 feet has a smaller area than a square with the same perimeter.
5. Use the cutting argument to show that an $a \times b$ nonsquare rectangle has a smaller area than a square with the same perimeter.
 6. Suppose you have 32 m of fencing to build a rectangular pen. You can build a bigger pen by building it against your barn, which is 25 m long, because the fencing is needed for only three of the sides. What dimensions maximize the area of this pen?
 7. Suppose you have 32 m of fencing and you want to build a rectangular pen against a not-so-long barn (one that's only 13 m long). Again you plan to use the barn as one side of the pen. What size rectangle maximizes the area of the pen now?

Can you relate this problem to something you already know?

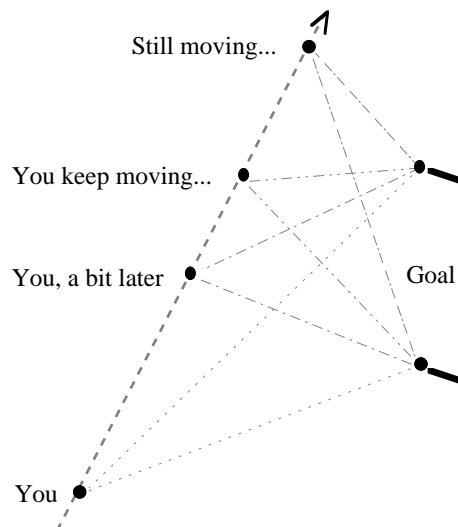
Geometry software is an ideal tool for experimenting with this problem.

We'll look at this problem again in Investigation 6.4.

- 8. Challenge** To score in soccer, one has to kick a ball between two goal posts. Suppose you are running straight down the field toward the goal, but off to the side because of the other team's defense. As you run, you have various openings on the goal posts. From what position should you shoot if you want the widest angle for getting the ball between the goal posts?



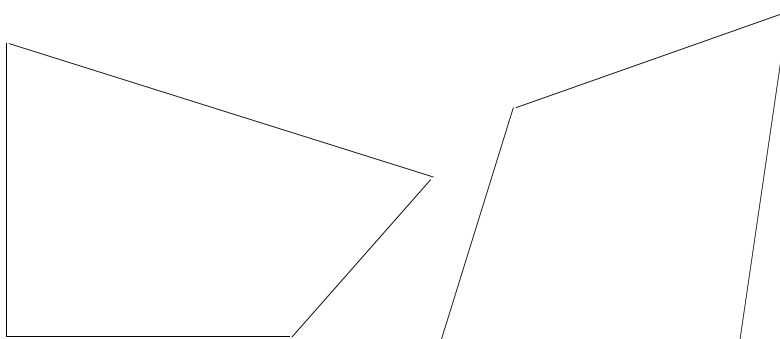
- 9.** What if you're playing soccer and you're moving the ball along some straight line that is not necessarily perpendicular to the goal line? Describe how to locate the spot that maximizes the angle in the picture.



CHECKPOINT.....

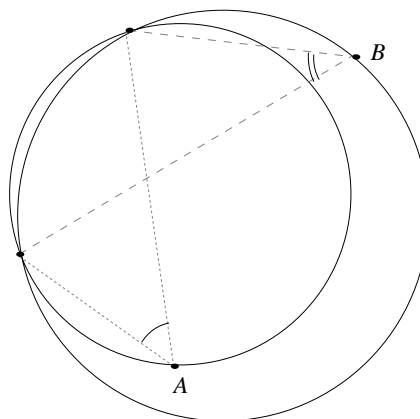
What about tracing and then cutting up the polygons and comparing the parts? What else might you do?

10. Of all the rectangles with a given perimeter, which one has the greatest area?
11. Which of these two polygons has the greater area? Find a way to convince your teacher or someone else that your answer is correct.



At least two angles are important—the one from the bottom of the picture to your eye to the top of the picture, and the side-to-side angle—and they maximize in different places.

12. You are in a gallery looking at a picture on the wall in front of you. Your eye level is 5 feet, the bottom of the picture is 5.5 feet above the ground, and the picture is 4 feet tall and 6 feet wide. How far from the wall should you stand for the maximum viewing angle? Is this likely to be the best place to stand?
13. Which of the two marked angles is larger? Why do you think so?



14. Explain why the hypotenuse is the longest side of a right triangle.

15. Define each of these words:

rectangle

square

perimeter

angle

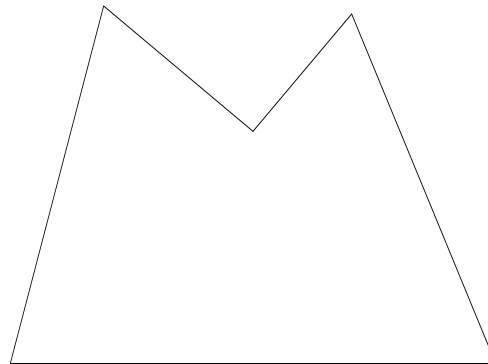
polygon

inscribe

16. **Write and Reflect** Make a list of all the strategies you have used so far for solving optimization problems. Illustrate each strategy with an example of a problem that can be solved with it.

TAKE IT FURTHER.....

17. Describe a way to create a polygon with the same sidelengths as the polygon below, but with greater area.



Imagine the vertices as hinges.

THEOREM 6.2

Of all the polygons having a given perimeter and a given number of sides, the regular polygon has the greatest area.

What does “regular” mean for a polygon?

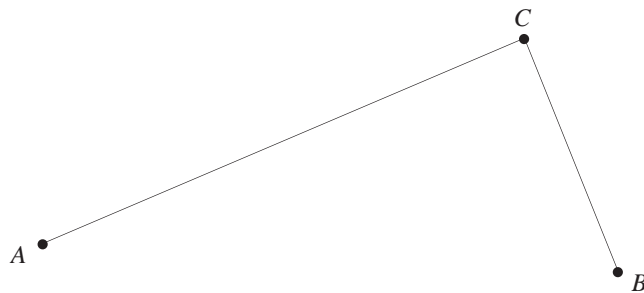
You'll prove Theorem 6.2 later. Right now you might try proving it for a special case: suppose the given perimeter is 120 ft and the given number of sides is 6.

You might want to use Theorem 6.2 again.

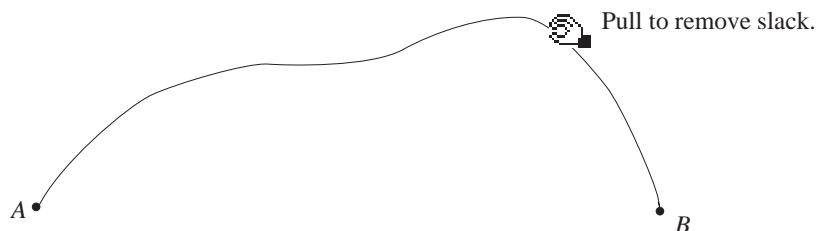
18. Solve the following generalization of Problem 6 using Theorem 6.2:

Suppose you have 840 feet of fencing and you want to build a four-sided pen (not necessarily a rectangle) against a very long stone wall, using the wall as one side of the pen and your fencing for the other three sides. What shape and dimensions maximize the area of the pen?

19. Suppose you have 840 feet of fencing and you want to build a *five*-sided pen against the stone wall, using the wall as one side of the pen and your fencing for the other four sides. What shape and dimensions maximize the area of the pen?
20. Compare the area of the pen you found in Problem 19 to the area of the pen you found in Problem 18. State your conjectures.
21. In the figure below, imagine that A and B are fixed, but C can move anywhere as long as $AC + BC$ stays the same. As you move C , $\triangle ABC$ changes shape. For what position(s) of C does the triangle have the greatest area? Why?



One way to model this situation is to take a piece of string—its length will represent $AC + BC$ —and pin its ends at A and B . Pull out the slack so that the string describes two line segments. The intersection of these two segments is one place where C can be.



You can do this as a thought experiment if there's no string available.

- 22.** In Problem 21, what kind of a curve do you get if you drag C around while keeping $AC + BC$ constant?
- 23. Challenge** Of all triangles of perimeter 24, which one has the greatest area? Justify your answer. Does your finding relate to any conclusions drawn in Problems 18–20?
- 24.** In Problem 4, you saw that of all the rectangles with the same perimeter, the square has the greatest area. Here is a related problem: Of all the rectangles of area 64, which one has the *smallest* perimeter?

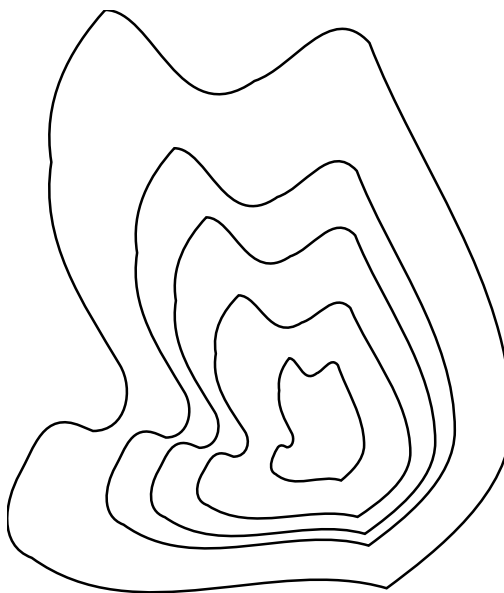
How is this a “related problem?”

CONTOUR LINES AND CONTOUR PLOTS

In this investigation, you will construct *contour plots*. Contour plots provide a new way for visualizing functions, and they can often be used to solve optimization problems. A topographic map is an example of a contour plot.

Have you ever seen a topographic map? A topographic map usually depicts land elevations and sea depths by curves or lines that represent points of about the same elevation:

All the points on the same curve are about the same distance above or below sea level. Is this a map of a mountain? A lake?



Although this diagram shows only closed curves, other maps may include open (partial) curves. It depends on the size and scale of the map.

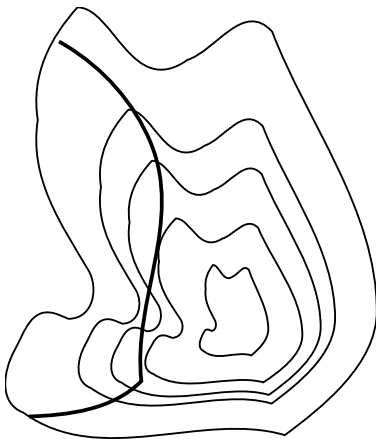
Contour plots are used extensively in mapmaking. The U.S. Geological Survey began looking for mapping volunteers in the summer of 1994, and may continue to need help updating the maps of your area. Volunteers can sign up for a specific map or a part of a map. Volunteers of any age and background are accepted; in particular, the USGS is looking for people with some hiking ability, basic mathematics skills (you have plenty of those), and good vision. It will be helpful if you can learn how to interpret contour lines.

What does *topography* mean?

Maps need to be updated because many changes can happen to the topography around you, sometimes due to things such as the construction of roads or buildings, changes in the course of a stream or river, the digging or filling in of quarries or mines, and the destruction caused by earthquakes or floods. In addition to the need for updated

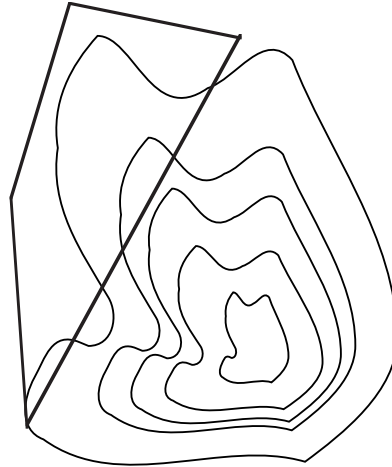
If you like being outdoors and you're interested in interpreting maps, why not give this a try?

See if people in your class can guess the place described by your map.



information on changes, many of the existing features on current maps must be verified. You can sign up as an individual or with a group of friends or relatives. Check your library for recent address and telephone information for the U.S. Geological Survey.

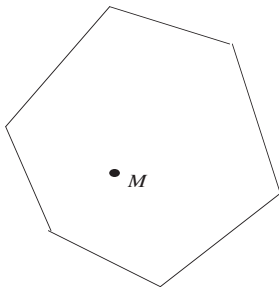
1. **Write and Reflect** Find at least two ways in which contour plots are used by weather forecasters. You may have to contact a local weather service for help.
2. Make a contour plot of your gym (with all the bleachers pulled out), your football stadium (including stands, if there are any), a local sports stadium (include the stands), an outdoor amphitheater (like Blossom in Ohio or Great Woods in Massachusetts), or an auditorium or music hall that you've visited.
3. Suppose that the picture in the margin is the topographic map for a mountain and that there is an increase of 1000 feet between contour lines.
 - a. If the lowest part of the mountain is 100 feet above sea level, sketch the places on the mountain that are about 1600 feet above sea level.
 - b. The exact peak is unmarked here, but given what you know about these contour lines, what is the maximum elevation that it could be?
 - c. What is the steepest part of the mountain? Explain.
4. If the dark curve on the contour map shows the path of some hikers on the mountain, what is the highest elevation they reach on their hike?
5. This same picture could be a map of a pond for which the outer contour line represents the edge of the pond and each inner contour line represents an increase in depth of 10 feet. Imagine that a camp owns the property bounded by the quadrilateral shown on the map on the next page and that the swim director wants to rope off a children's swimming area with water no deeper than 5 feet. Trace the map on page 29 on another piece of paper and sketch a proposal for such a roped-off area.



Why would people use the words *contour* and *level* to describe these paths?

The individual curves in a contour plot are called *contour lines* or *level curves*. A more precise definition of a contour line will come up soon, but for now you can think of a contour line as a curve that shows where a particular feature of a situation is invariant. When the contour lines make shapes that you recognize (circles or polygons, for example), you may be able to use the geometry of those shapes to solve optimization problems. For example, recall Problem 16 in Investigation 6.2.

The picture looked like this:



You are at M in a strange swimming pool with many sides. How could you find the shortest way out of the pool? Would you go to a corner or a side?

One student, Pam, thought about doing it this way:

Wait until the water in the pool gets very calm. Then take your fist and pound the water, right by your waist. Ripples will go out from where you hit the water, and they'll be circles. Swim to the first place where a ripple touches a side of the pool.

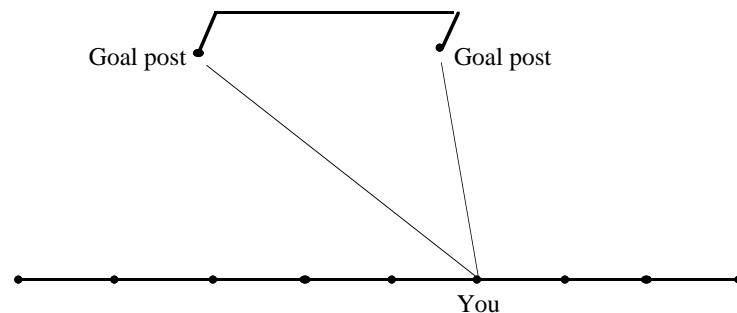
6. Draw a picture that shows what Pam is talking about. Why does that method work?
7. In Pam's method, the ripples are contour lines. Exactly what is constant for all the points on the same ripple?

8. Each ripple is an expanding circle. Even after it has expanded so much that it touches a wall, it continues to expand. Draw a sketch to show what it looks like a short time after touching a wall.
9. The ripple you've just drawn should intersect the wall in two places. Why are those two points of intersection not the best places to which to swim?

A CONTOUR PLOT FOR SOCCER

Remember the previous soccer problems? Suppose that you are on a team, and your coach wants you to learn how to find the best kicking spots for scoring a goal. This is a complicated skill to learn, so first you look at some simpler cases.

10. The coach draws a line straight across the field, lines up the whole team on that line, and then asks some questions:



The player's position is the vertex of the "kicking angle" and each goal post is on a side of the angle.

- a. Which player has the maximum kicking angle?
- b. How does the angle vary as you cross the field?
- c. How many players on the line have a kicking angle that is, say, 5° less than the maximum? Approximately where are they?
- d. Are there any players on the line with a kicking angle of 1° ? Approximately where are they?

This problem is more realistic than Problem 10 because players can kick from anywhere on the field.

11. At the next drill, let's say that everyone agrees that the player in the center has the best angle (let's say it is 40°). Now the coach asks, "Are there any other places on the field where you can kick with a 40° angle?" Place yourself in one of those positions. Sketch where the players could stand. (Your sketch will show a *contour line* for a kicking angle of 40° .)

- 12.** The team developed a contour map for the kicking angles over the *whole* field. Make a sketch to show what this map could have looked like. Show both goal posts and several contour lines. Are the contours straight, curved, irregular, or circular? Explain.

As you move along any contour line in Problem 12, the kicking angle stays constant. However, the contour map all by itself doesn't tell you what that constant angle is. Label the contour lines in your map with the measures of the kicking angles. Make sure to include the 90° contour line.

The number attached to a contour line is also called its *value*.

- 13.** How do the numbers behave as you move away from the goal in any direction? If you add a contour line between two existing ones, what can you say about the number attached to it?
- 14.** Suppose a player with the ball is crossing the field in a straight line, but not parallel to the goal line. Draw such a line on your contour map. Explain how the map can be used to determine the spot on the line from which the player has the largest kicking angle.

In a real game of soccer, you wouldn't be constrained to run in a straight line, nor would you care about the *best* angle, as long as it was good enough. And, depending on your strength and kicking accuracy, you might be willing to sacrifice the width of the angle a bit if it gave you a more open shot. But, the size of the angle can be one important consideration.

It would hardly be a favor to your team to stop dead in your tracks, work out the best angle, and then run on. Experienced soccer players just "see" the right place and do their best. But what does experience teach you? It teaches you when the current angle is *not* the best; it also teaches you whether going forward or backward will improve it.

- 15. Write and Reflect** Invent another problem that uses some of the same mathematics as the soccer problem.
- 16. Write and Reflect** As we look further into contour lines, it will be helpful for us to use the idea of *function*. What is meant by the word *function*? How have you used the function idea in mathematics? Is the word ever used in ways that are different from its mathematical meaning?

WHAT IS A CONTOUR LINE?

So far, contour plots have been used like this:

The nature of the problem assigns a number to each point on the plane. The number might represent temperature, barometric pressure, height above sea level, depth of a pond, distance from a swimmer in a pool, measure of the kicking angle on a soccer field, or a value of many other types.

A *contour line* is a set of points (a curve of some type) whose assigned values are all the same. A *contour plot* is a collection of contour lines with enough information to let you reason about the values of points in the regions between the lines.

..... **WAYS TO THINK ABOUT IT**

The process of assigning a number to each point on the plane is an example of what's known as a mathematical *function*. We used this term in the "Ways to Think About It" just before Problem 4 in Investigation 6.2. One way to think about a function in this new context is to imagine a point P moving around in the plane carrying a little calculator on its back. The calculator is programmed to always do the same calculation (to measure the kicking angle, for example), and that calculation depends solely on P 's current location. Sometimes there are many different points that cause the calculator to produce the same number (or *value*, as it's called). A contour line for the function is just the set of all points for which the calculator produces the same value.

The next problem asks you to experiment with such a function.

.....

17. Using geometry software, define three points A , B , and P , and consider the function f that calculates the sum of the distances $PA + PB$. Ask your software to measure $PA + PB$. (Remember that here PA means the distance from P to A , not the product of P and A .) Then, as you drag P around, the measure will change, showing you the particular sum (the *value* of your function). On

a transparency taped over the screen, mark enough points to let you draw a few contour lines for f .

Write a paragraph describing how you imagine this function f in your own head.

Let's make this a little more formal now. We can name a function, calling it, say, g . Then the value of g for any point P is usually written $g(P)$.

$g(P)$ is pronounced “ g of P ” and means “the *value* of g at P .” If you think of g as a calculating machine, $g(P)$ is the number produced by the machine when you “feed it” the point P , or it is the *output* of the machine g that you get when the point P is the input.

One sometimes asks, “What number does g assign to this particular position for P ?” One can also ask, “At what positions for P does g produce this particular number?” The set of all positions for P that cause a particular value for g , such as 5 or 8, is a *contour line* for g corresponding to that number. A *contour plot* for g is a collection of g 's contour lines (each corresponding to different numbers).

18. Suppose C is some fixed point on the plane, and suppose g is the function that takes a point P in the plane and calculates its distance from C , that is, $g(P) = PC$.

- a.** Make a contour plot for g .
- b.** What shape is each contour line? Give a proof.
- c.** This contour plot would give more information if each contour line were labeled with a number that told the value of g . Label each contour line with the appropriate number.

19. Suppose ℓ is some fixed line in the plane, and suppose f is the function that calculates a point's distance from ℓ .

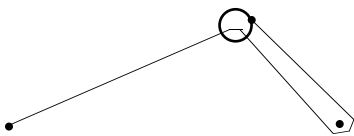
- a.** Make a contour plot for f .
- b.** What shape is each contour line? Give a proof.
- c.** Label each contour line with the appropriate number.
- d.** What would the contours for f look like if you were looking for all points in *space* (not just in the plane)? That is, what is the set of points in space that are a fixed distance from a given line ℓ ?

You might want to use your geometry software or a piece of string.

That is, $f(X) = XA + XB$. To find the value that f assigns to a point X , measure the distance from X to A , measure the distance from X to B , and then add.

What do people mean by the *range* of a function?

A gadget like this might help:



Try to invent a gadget using string that will help you draw the contour lines for j .

- 20.** This is an extension of Problem 17. Put two points A and B on the plane. Let f be the function that assigns $XA + XB$ (a number) to X (any point). Make a contour plot for f .
- Have you seen curves like these contour lines before? Describe them as precisely as possible.
 - Label each contour line with the appropriate number.
 - Of course, your contour plot cannot show *every* contour line. Counting your smallest contour line as “the first,” imagine drawing a new contour line between your second and third contour lines. What are the possible numbers that could belong to that new contour line?
 - Draw a straight line across your contour plot. Explain how to locate the point on that line for which f produces the smallest value.
 - On your contour plot, draw a circle that contains A and B in its interior. Explain how to locate the point or points on the circle for which f produces the smallest value.
 - Can any number you pick be a value for *some* contour line? If so, explain why. If not, give an example (a number that couldn’t possibly be a value for f) and explain why not.
- 21.** Suppose that, E and F are fixed points in the plane, and that f is a function defined so that $f(Z) = ZE + 2ZF$ for any point Z .
- Make a contour plot for f . Label it appropriately.
 - Draw a straight line on your contour plot. Explain how to locate the point on the line that minimizes the value of f .
- 22. Challenge** Suppose A , B , and C are three fixed points in the plane and j is a function defined to measure the sum of the distances from a point P to A , B , and C :

$$j(P) = PA + PB + PC.$$

Make a properly-labeled contour plot for j .

23. Suppose that A and B are fixed points, and that P is free to roam anywhere in the plane (on both sides of \overleftrightarrow{AB}). Suppose h is a function defined as

$$h(P) = m\angle APB.$$

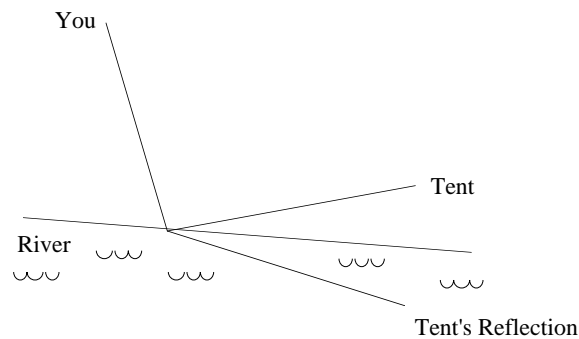
You could use color-coding to explain certain aspects of your map.

- a. Make a properly-labeled contour plot for h .
 - b. Are there any points in the plane which seem to be on *every* contour line in your plot? Are there any points that couldn't be on *any* contour line in *any* contour plot for h ? (That is, are there any points on the plane for which h produces *no* number?) Explain.
24. In Problem 23, what is the contour line that P traces if P is above \overleftrightarrow{AB} and $m\angle APB = 90^\circ$? Describe how $m\angle APB$ behaves if P is inside this contour line (and above \overleftrightarrow{AB}). If P is outside?
25. **Write and Reflect** Explain how to draw a contour plot for a function that is defined on the points in a plane. Illustrate with examples.
26. **Write and Reflect** Explain how a contour plot can sometimes help you solve an optimization problem. Use an example in your explanation.

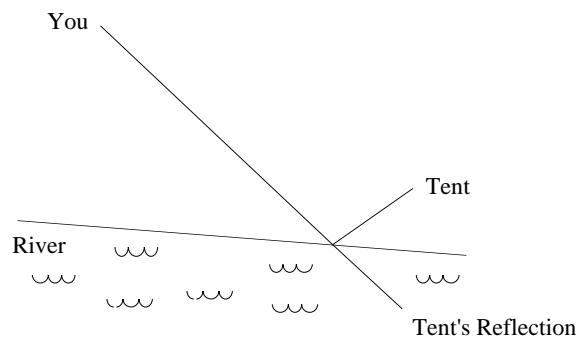
REVISITING THE BURNING TENT

Returning from a hike while on a camping trip, you see that your tent is on fire. Luckily, you're holding a bucket and you're near a river. Where along the river should you fill the bucket with water to minimize the total distance you travel to get back to the tent? (See Problem 8 in Investigation 6.2.)

When you first looked at the burning tent problem, you might have analyzed it using the idea of symmetry and reflection. When you look into a mirror, an object's reflection appears to be the same distance from the mirror as the real object is, but on the opposite side of the mirror from the real object. This makes the images and the real scenes that cast the images symmetric with respect to the mirror.



Suppose there is a mirror along the river bank. The path from You to the river to the Tent's Reflection is the same length as the path from You to the river to the Tent.



Clearly, the shortest path from You to the river to the Tent's Reflection is straight, and so

That tells us where the best spot is, but we can also get insight into the problem by looking at some “wrong” answers. Let’s pick a non-optimal spot along the line, like the one pictured in the first of the two figures above. Surely, there are other spots P (not necessarily on the river’s edge) that are exactly as bad as that spot—no better and no worse.

A set of points that satisfies a set of conditions is sometimes called a *locus*. Find out how the words *locus*, *location*, and *locale* are related.

Think “string.”

27. What about those spots (*not* along the river) that are just as bad (an equal total distance from You and the Tent) as some particular non-optimal point along the river? That is, what is the set of points some constant *total* distance from You and the Tent? Pick a total distance and perform an experiment to determine the contour line for that value.
28. Devise an efficient way to draw contour lines like the one in Problem 27.

29. Devise a construction technique in your geometry software that allows you to make contour lines like the one in Problem 27.

.....
WAYS TO THINK ABOUT IT

Here's an outline of a way to use geometry software to locate points in a plane that are a particular (total) distance from *two* points A and B .

Decide on the total distance (it must be greater than AB) and construct it. A line segment will do well. Let us call this distance k .

Divide that total distance into two parts, which we'll call d_1 and d_2 . A point (call it P) on the line segment will do.

Create a circle of radius d_1 with center point A and a circle of radius d_2 with center B . Any points that are on *both* of these circles are therefore (If there are *no* points of intersection between these two circles, what must you change?)

As P is moved, it changes the values of d_1 and d_2 without changing their total. Tracing the intersection points of the two circles will produce an ellipse. (If the circles don't intersect, increase k .)

.....

The contour lines we've been drawing are called *ellipses*:

DEFINITION

Begin with two points A and B and a positive number k . The set of all points P for which $PA + PB = k$ is called an *ellipse*. A and B are called the *foci* of the ellipse (A is one *focus* and B is the other *focus*).

So we can define an ellipse as a contour line:

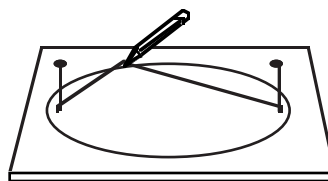
DEFINITION

An *ellipse* with foci A and B is a contour line for the function f that is defined by the calculation

$$f(P) = PA + PB.$$

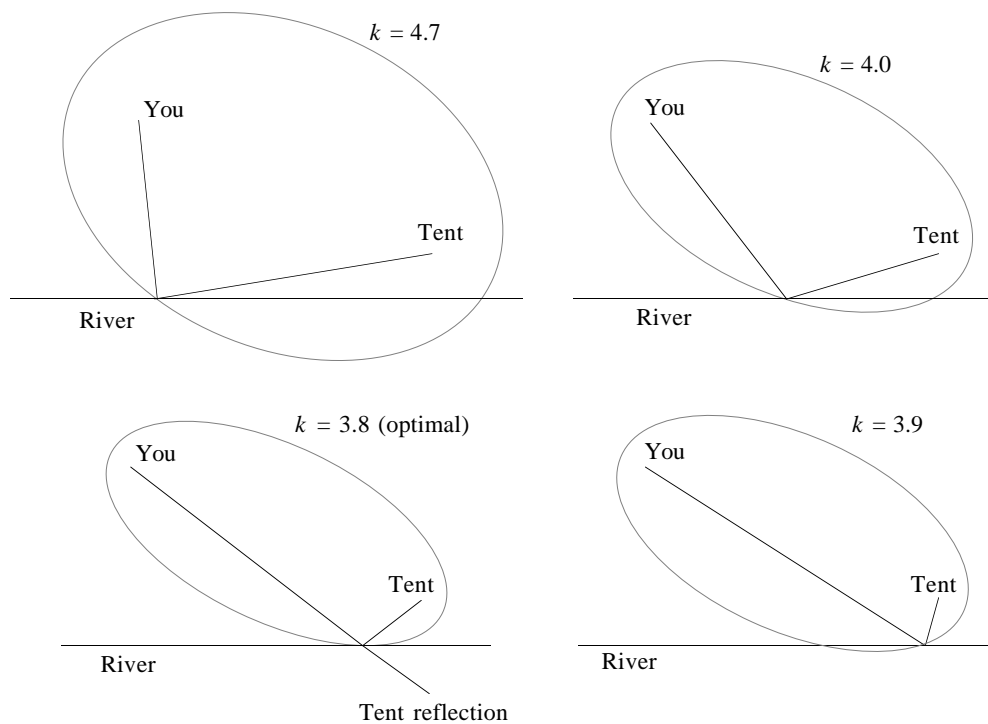
Two definitions are *equivalent* if they define the same object.

30. Are these two definitions of an ellipse equivalent? Explain.
31. When builders need to construct elliptical shapes, they use a method like that illustrated in the picture below: They drive nails in at the two foci of the ellipse and then tie each end of a piece of string around the nails. With a pencil, they trace around the nails, keeping the string taut.

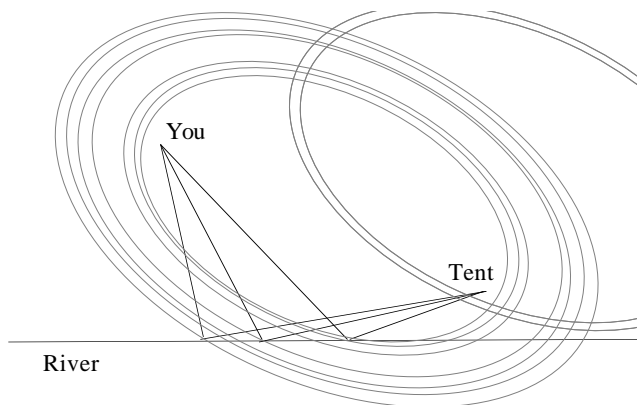


Explain how this setup guarantees that all of the points drawn by the pencil are the same total distance from the two nails.

Here is a set of drawings that we made for the burning tent problem. Each one shows some path from You to the river to the Tent, and a curve containing all the other points on the plane (not just on the river) that are the same total distance from the Tent and You.



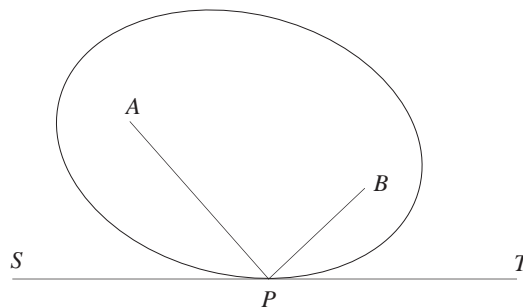
Here is a picture with many of the contour lines superimposed—a contour plot for the burning tent problem.



Even though people say “contour line,” the word “line” doesn’t mean “straight line”—contour lines are often curves.

32. Notice how these contour lines relate to each other and to the optimal spot on the riverbank. Explain how the contour lines can be used to solve the burning tent problem.
33. Explain how you could use the string-and-nails setup to solve the burning tent problem.
34. Make a sketch in your geometry software that shows how ellipses can be used to solve the burning tent problem. Check that the solution from your contour-line method agrees with that from the reflection method discussed in Investigation 6.2.
35. **Challenge** Ellipses are interesting curves that have many wonderful properties. For example, the celebrated “tangent property” for the ellipse says:

Suppose you have an ellipse with foci A and B , and you draw tangent \overline{ST} to the ellipse at P . Then $m\angle SPA = m\angle TPB$.



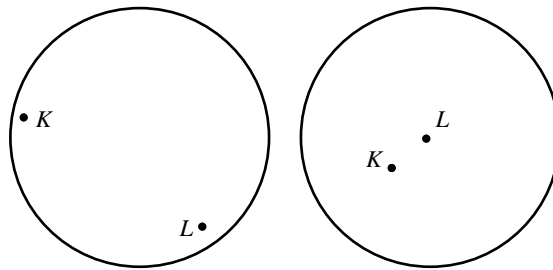
What does *tangent* mean?

Think of A as *You*, and B as the burning tent.

This is also why the focus is called a “focus.”

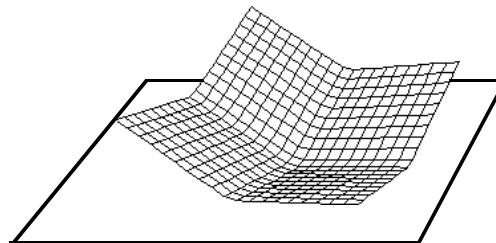
Suppose you place a ball at each focus of an elliptical billiard table and shoot one of the balls in *any* direction. The tangent property implies that, if you hit it hard enough, it would hit the other ball after just one bounce off the elliptical cushion without even aiming! Prove that the “tangent property” stated above is true for any ellipse.

36. Natasha is in a circular swimming pool at K . She wants to swim to L but first she wants to swim to the edge of the pool to leave her sunglasses. Explain how to find the best place to put the sunglasses to minimize the total amount of swimming.



Here are two possible arrangements for this problem.

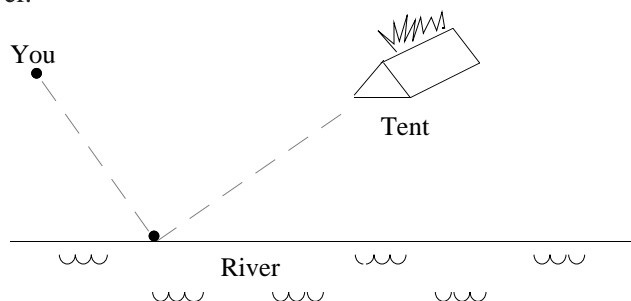
37. a. Investigate topographic maps and relief maps; describe how they are different.
- b. Suppose O is a fixed point and f is defined so that $f(X) = OX$. What would a relief map for f look like?
- c. Let A and B be fixed and g be defined so that $g(X) = AX + BX$. Describe g 's relief map.
- d. Below is a depiction of a three-dimensional relief map for a function. What might the contour plot look like?



In mathematics, relief maps are called *surface plots*. Surface plots are supposed to be three-dimensional; what you see in this module are two-dimensional drawings of surface plots.

Look at Problem 21 in Investigation 6.3.

38. Suppose you're on (yet another) camping trip and your tent is (yet again) on fire. Luckily (again) you have a bucket and you're near a river. Where should you get the water to minimize the *time* it takes to get back to your tent? Justify your answer.



CHECKPOINT.....

39. Define the following words:

contour line

contour plot

function

ellipse

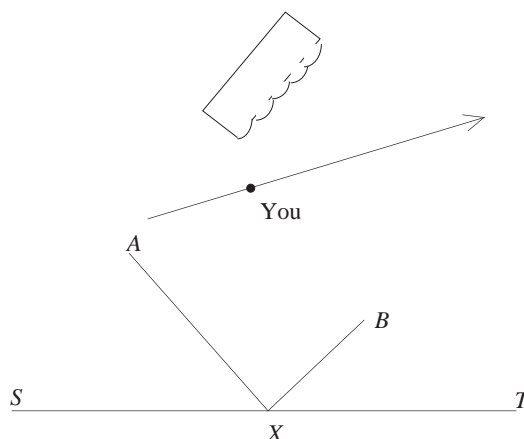
focus

tangent

What does *façade* mean?
How is it pronounced?

In “real world” problems, things are rarely crystal clear. Here, there are several possible meanings for “best”: nearness, widest angle, least perspective distortion, personal taste Think about (and perhaps discuss) the possibilities and make (and explain) *your* decision.

40. Imagine that you’re driving through the country, and that the views are so beautiful that you hardly mind the relentless straightness of the road. A building to your left, which was quite plain in appearance from the rear, has a beautiful front *façade* that you are eager to photograph. You figure that you are now about as *close* to the building as you will ever be on this road, but because the building is not set parallel to the road, perhaps you’d get a better *angle* farther along. What *is* the best viewing position along the road?



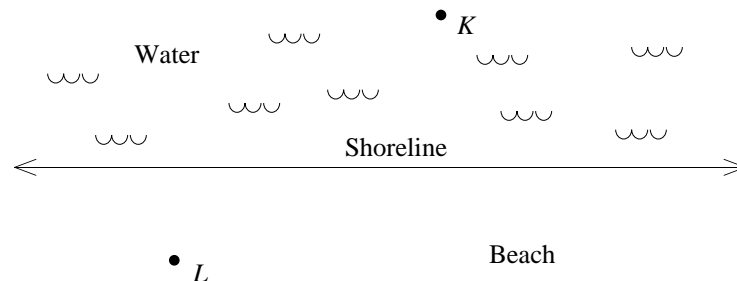
TAKE IT FURTHER.....

41. Do you remember Problem 7 in Investigation 6.2? “You’re in a rectangular swimming pool at K , out of reach of the sides of the pool. Before swimming to L , you want to swim to a side of the pool to put down your sunglasses.” Explain how to use contour plots to find the place to put down your sunglasses in order to minimize your swimming.

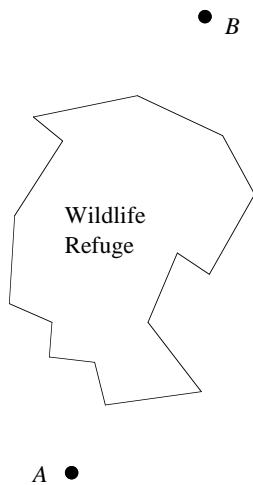


Some people can swim faster than they can run.

42. Rafael was lounging on the beach at L , thinking about how to minimize the total time it would take him to run to the shore and swim out to a raft at K . He could run twice as fast as he could swim; where should he hit the shore to minimize the *time* it takes to get to the raft?



43. Here is a problem and a proposed answer. Read both carefully. Then critique the problem and support or refute the answer. (Note: there are some ambiguities in the problem and some weaknesses in the answer.)



Problem: Two cities A and B , separated by a large wildlife refuge, decide to pool their resources to build a recreation center. By agreement, the new center cannot be located in the refuge, nor can the traffic generated by the center cross the refuge. New roads must be built, and the cities want to find a location that minimizes that cost. Where should the recreation center be placed?

Answer: Clearly, to do the *least* roadbuilding to connect the two cities with the same point, we would have to build the roads along the line segment \overline{AB} . But we can't, because that would cross the refuge, so the total roadbuilding must be some length greater than AB . All the locations with that very same total distance from A and B lie along an ellipse, so the best places to locate the recreation center would be anywhere on the smallest ellipse that has A and B as foci and does not cross the refuge.

44. **Challenge** Suppose ℓ is a fixed line, E is a fixed point not on the line, and f is a function that assigns a numerical value to each point P this way:
- f finds the distance from P to E ;
 - f also finds the distance from P to ℓ ;
 - f adds the results and assigns *that* number to P .

Make a contour plot for f . Label your contour lines with the appropriate values calculated by f .

This is a 3D version of
Problem 18.

45. Suppose you have a function g that calculates the distance between a point P in three-dimensional space and some fixed point O . What would a picture analogous to a contour plot look like for such a function?

1. Write and Reflect Discuss the techniques you used to solve problems in the previous investigations in this module. How did you systematize what you knew about a problem? How did you visualize alternative pathways? What ideas helped you combine your experimental data with logical deduction? How did you invent new theories? What helped you reduce a complicated problem to a more manageable, but equivalent, problem?

2. Define each term:

conjecture

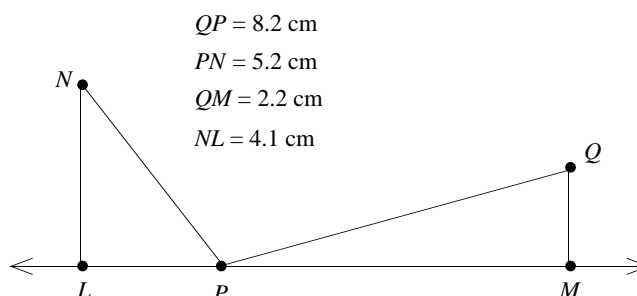
theorem

equidistant

vertex (or vertices) of a triangle

midpoint of a segment

3. In the following figure, \overline{NL} and \overline{QM} are perpendicular to line \overleftrightarrow{LM} .



- What is the measure of $\angle NLP$?
- What is the distance from Q to \overleftrightarrow{LM} ?
- What is the length of \overline{LP} ?
- Is P equidistant from N and Q ? Explain.
- Is \overline{NL} parallel to \overline{QM} ? Explain how you know.
- If P were free to move to the right along \overleftrightarrow{LM} , what would happen to NP ?

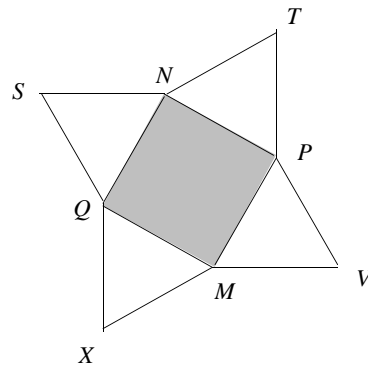
4. Now assume that in Problem 3, P is a point that is free to move along \overline{LM} . Evaluate each of the following statements as true or false. Correct any false statements.
- No matter where P moves on \overline{LM} , $NP + PQ < NQ$.
 - No matter where P moves on \overline{LM} , $NP + PQ$ will be the same.
 - If P is at the midpoint of \overline{LM} then $LP = MP$.
 - If P is at the midpoint of \overline{LM} then $NP = PQ$.
 - Where P is currently located, $NL + LP = NP$.
 - If $\angle NPQ$ is a right angle, then $NP = PQ$.
 - If N' is the reflection of N across \overline{LM} , then $m\angle NPL = m\angle N'PL$.
5. Swimming lessons are being given at a local pool this summer. To get the Swimmer's badge, a student must swim the length of the pool under water. This is the last test:

Dive into the pool and swim under water until you get to the other end of the pool. You must touch the bottom of the pool once during the swim.

Of course, the swimmers want to swim by the shortest possible route, especially the ones who aren't sure how long they can hold their breath. Carol, who is a careful thinker, takes a starting position, then dives in. The pool is 50 feet long, 20 feet wide, and 8 feet deep. Draw a diagram of the pool showing where Carol must have stood to start, and the best path under water. (Assume a constant speed for the entire path.) Explain why this is the best path. How far did she swim? At what angle did she dive into the pool? At what angle did Carol emerge?

6. In studying the burning tent problem, many students make this false conjecture: "It doesn't matter where you stop at the river; the distance $AP + PB$ will always be the same." Write an explanation or proof of why this is false. How can we be sure that the total distance ($AP + PB$) will *not* be the same everywhere along the river, and that, in fact, there *is* some point P which minimizes the total travel $AP + PB$? Your explanation must use logic and mathematics to show why this is true. Simply measuring some number of points is not a sufficient response, although you may want to use the information you gain from doing so in your explanation.

7. Given any $\triangle ABC$, let M_1 be the midpoint of \overline{AB} and M_2 be the midpoint of \overline{BC} . Find the shortest path from M_1 to \overline{AC} to M_2 . Label the point where the path hits \overline{AC} as X . Will X always, sometimes, or never be the midpoint of \overline{AC} ? Explain your answer.
8. A farmer owns a square farm and wishes to buy adjacent farms for her children. Because of highways and natural borders such as rivers, the adjacent farms are triangles. (To keep our examination of the situation simple, let's assume the triangular farms are equilateral and congruent.) See the diagram shown here, and answer the questions that follow.

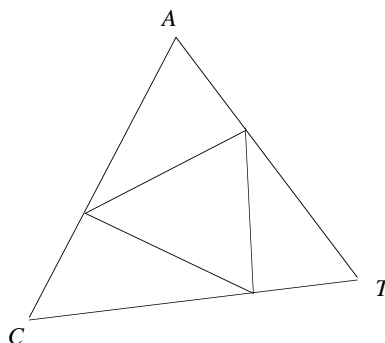


- a. If another square were drawn adjacent to the first farm so that it shared side \overline{NP} , would point T be inside, outside, or on the border of the square?
- b. Assume that the perimeter of the first farm $NPMQ$ is 8 km.
- What is the area of the first farm?
 - What would be the perimeter and area of each child's farm?
 - What would be the total perimeter and area of the land owned by the entire family?
 - What is the maximum land area the farmer could acquire if she could buy as much land surrounding her farm as possible, given the restriction of a total perimeter no greater than that just calculated in part b iii? Assume there are no restrictions on the shape of her children's farms.
 - What would be the shape of the family farm then?

- c.** Write a paragraph or two about why, if you were the farmer, you might choose the configuration shown on the previous page in expanding your farm to include land for your children, or why, given your calculations, you might choose another shape for the land and how you would divide it among your four children.
- 9.** What if the farmer's main concern is not the shape of the children's farms but the cost of fencing? Let's assume that the first farm is still square, and that the farmer will purchase four adjoining plots of land for her four children. The children's farms can be any shape, but they must all be the same shape, and each one must have an area equal to $\frac{1}{4}$ the area of the original farm. What shape should the farms be now to minimize the amount of fencing around the entire family farm? Explain and justify your answer.
- 10.** Drink boxes are a fairly new type of packaging. A typical boxed juice contains about 250 milliliters (ml) and measures $10.5 \text{ cm} \times 6.5 \text{ cm} \times 4 \text{ cm}$. What is the volume of this box in cubic centimeters?
- a.** Is this the best box size? Suppose that you are the manufacturer, and you want to maximize your profit by reducing the packaging costs. What size rectangular drink box will hold 250 ml and use the least cardboard?
- b.** Design a drink container (of any shape) that would hold 250 ml of juice, using the least cardboard. (Be sure that it could be held comfortably by a child or an adult and could be packaged, shipped, and stacked on a store shelf.) Explain your choice with enough information to convince a packaging engineer that this is the best shape.
- 11.** Draw a circle and a line tangent to the circle. Which point on the tangent line is closest to the center of the circle? When you draw a radius from the point of contact to the center of the circle, how are the radius and the tangent line related?

What does it mean to *inscribe* a polygon in another polygon?

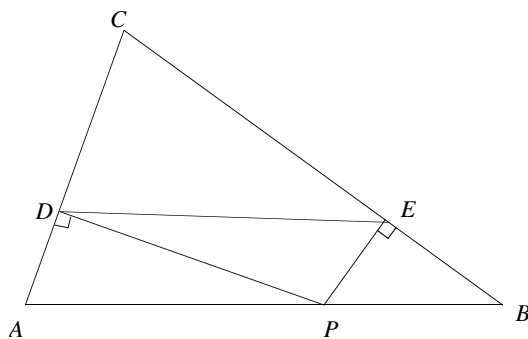
12. Take an acute triangle and *inscribe* another triangle in it. Here's one example:



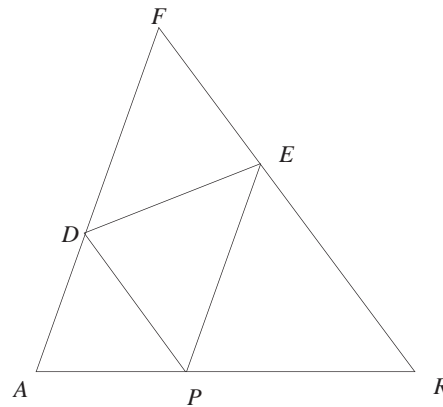
This is a type of minimal path problem.

Of all the possible triangles inscribed in $\triangle CAT$, which has the smallest possible perimeter?

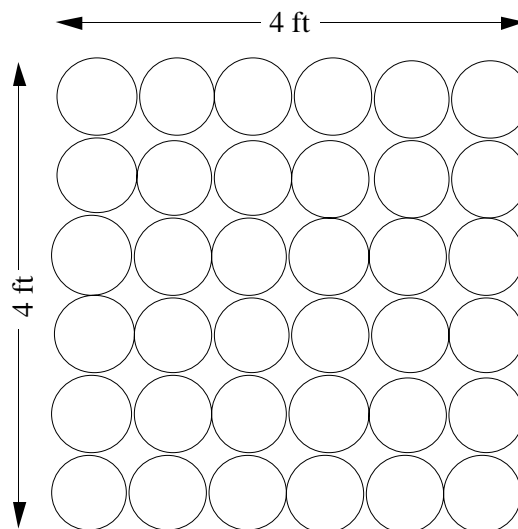
13. In $\triangle ABC$, P is a point on \overline{AB} , $\overline{PD} \perp \overline{AC}$, and $\overline{PE} \perp \overline{BC}$. Where should P be located on \overline{AB} in order to make the length of \overline{DE} as small as possible?



- 14. Challenge** In $\triangle ARF$, P is a point on \overline{AR} , $\overline{PD} \parallel \overline{RF}$, and $\overline{PE} \parallel \overline{AF}$. Where should P be located on \overline{AR} in order to make DE as small as possible?



- 15.** Cut wood is often sold by the *cord* and the *half-cord*. A cord is a stack that measures 4 feet \times 4 feet \times 8 feet. The wood can be bought in logs which are 8 feet long (“log lengths”), cut into smaller logs, or cut and split into easy-to-manage pieces. For firewood, many people buy a cord or half-cord of cut and split logs. Paper and lumber mills, however, often purchase wood in log lengths.

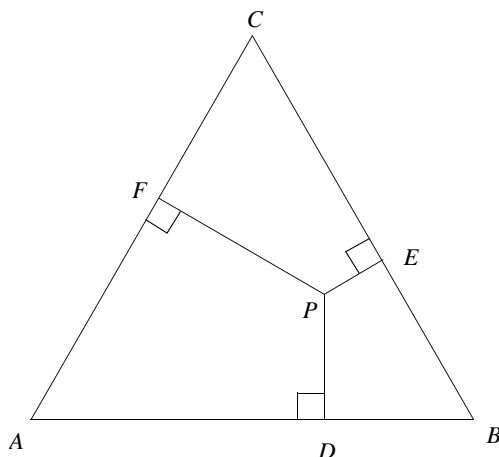


This picture shows one way to stack a cord of log lengths.

Of course, wood doesn't come this way; there are many different diameters in a pile. But this is a way to start thinking about the problem. Should Jamal buy one big log? Many small logs? Does it matter that small logs burn faster than large ones?

Suppose that, when Jamal orders a cord stacked like this, he may choose the diameter of the logs. What diameter should he pick if he wants to maximize the amount of wood in the stack and minimize the amount of air he buys?

16. Point P moves around the interior of equilateral $\triangle ABC$. From P , perpendiculars \overline{PD} , \overline{PE} , and \overline{PF} are drawn to the sides of the triangle. Where should P be located to make the sum of the distances to the sides of the triangle (that is, $PD + PE + PF$) as small as possible?



INTRODUCTION TO THE MAGICAL MIX

Who would ever have thought there were so many different ways to define “the shortest path from here to there?” What is the shortest path from point A to point B ? What is the shortest path from point C to line ℓ to point E ? Or what is the shortest path from Fall River to Grand Rapids, if you have to stop in Houston and Juneau and Kathmandu along the way?

In solving optimization problems such as those in Investigations 6.1–6.5, you probably found it necessary to use a lot of the geometry you know and to learn a lot more. Properties of triangles, circles, polygons, angles, ways of finding distance, reflections and rotations of objects, even trigonometry: all of these topics could have come up as you worked your way through optimization problems. But just as important as the *content* are the *methods*, the mathematical habits of mind you develop to deal with these problems. For example, how did you systematize what you knew about a problem? How did you visualize various alternative pathways? In what ways did you combine your experimental data with logical deduction? How did you invent new theories? In what ways did you reduce a complicated problem to a simpler, more manageable but similar problem?

A year from now, you may not remember exactly how to solve every problem you worked on, but if you’ve developed a mathematical way to look at things, you will always be able to reconstruct solutions.

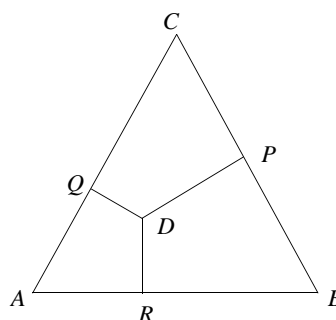
In studying mathematics, you solve problems and learn mathematical facts and relationships. But you *also* learn how to create, invent, conjecture, experiment, and prove. All of that is just what you’ll do in this section of the module, where you’ll explore many aspects of one problem. We’ll also look at one high school student’s clever way of thinking about it.

A STUDENT OUTSMARTS THE TEST

Some years ago, a student of one of the authors faced this problem on a standardized, multiple-choice achievement test:

PROBLEM

Given an equilateral triangle of sidelength 10, and a point D inside the triangle, what is the sum $DR + DQ + DP$ of the distances from D to the sides of the triangle?



This student's name was Rich. From then on, in the spirit of professional mathematicians, the students in that class referred to the function that calculates the sum of the distances from an interior point to the sides of an equilateral triangle as "Rich's Function."

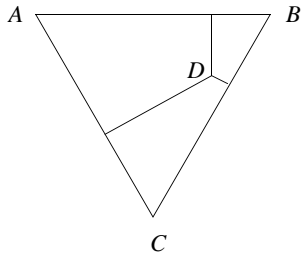
- A. 10
- B. $5\sqrt{3}$
- C. 30
- D. 8.6

Rich had no ruler and no computer. See what you can do with the problem without those tools.

1. Before reading further, think about this problem and see if you can find a strategy for solving it. (Remember that the distance from a point to a line means the distance along the *shortest* path. In the diagram, this means that \overline{DR} is perpendicular to \overline{AB} , \overline{DQ} is perpendicular to \overline{AC} , and \overline{DP} is perpendicular to \overline{BC} .)

Perhaps the writers of the test question above intended to find out if the students knew a particular theorem. As it turns out, our student didn't know it. But Rich either outwitted the test or demonstrated something else the testers may have been looking for: the ability to explore and solve a problem through the mixing of deduction, experimentation, and reasoning by continuity. Here's how he went about it:

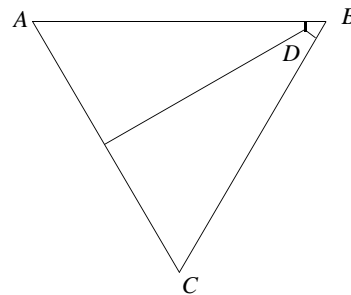
What does *deduction* mean?



D could go almost anywhere.

First, Rich decided that *one* of the four numbers listed had to be right because there was no choice like “not enough information.” Then he tried to imagine placing D at several different spots inside the triangle in order to compare the sums of the distances. Yet there were no tools with which to measure anything during the test. That’s when Rich realized that the location of point D within the triangle must not matter; in fact, the problem said nothing specific about where inside the triangle D is located. This was crucial: that *no matter where D is placed* inside the equilateral triangle, the sum of the distances from D to the sides of the triangle must be the same.

By drawing this conclusion, Rich got another good idea: imagine moving point D close to a vertex to see what that does to the three distances.



Right next to a vertex, two of the shorter distances were almost zero, and the third distance was almost the height of the triangle. Therefore, the answer must be the height of the triangle, which is easy to find in an equilateral triangle. So Rich came back to class after that test with the following conjecture:

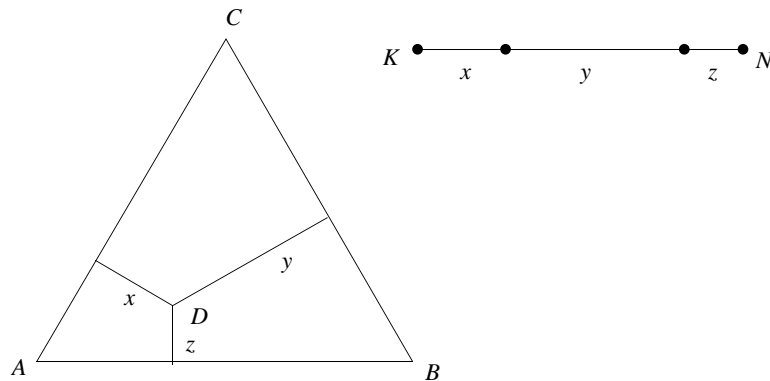
The sum of the distances from *any* point inside an equilateral triangle to the sides of the triangle is equal to the length of an altitude of the triangle.

- 2. Write and Reflect** Teachers who looked at this clever solution said that the student used “reasoning by continuity.” What do you think that means?
- 3.** If you have not already found the sum of the distances to the sides of the triangle in Problem 1, do so now by using Rich’s reasoning, what you know about equilateral triangles, and the Pythagorean Theorem. See if Rich’s reasoning leads to one of the answers given on the multiple-choice test.

These facts will help: the triangle has sides of length 10, and the altitude of an equilateral triangle bisects the base.

Rich made the conjecture about the altitude of the triangle during the test, under pressure, and didn't have time then to prove whether the conjecture was true. But, as it turned out, the method of experimenting with the position for D and of looking at unusual spots or extreme cases helped solve the problem. The class spent the next couple of days discussing the student's insight and the kind of mathematical thinking that led to it. Students wondered whether they could *prove* the conjecture and turn it into a theorem.

4. Before going on to the proof of the conjecture, use geometry software to make a sketch and a dynagraph of the situation as pictured below. (Save both for use in future explorations.)



The dynagraph allows you to see the contributions of each length to the total sum of the distances x , y , and z . As you move D around, think about the following questions: What happens to each individual length? What happens to the *sum* of the lengths?

5. Design a *mechanical device* (you can use string, nails, washers, weights, and so on) that lets you experiment with a model of the situation.

Reading proofs is one way to begin learning how to make your own proofs.

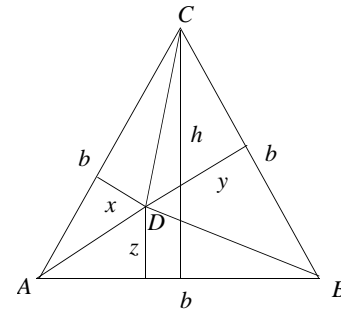
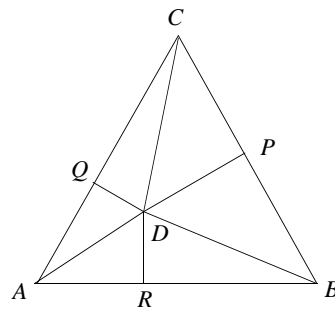
A common geometric habit of mind: Calculate area in more than one way.

Here is one way to prove the conjecture. In this proof, the area of $\triangle ABC$ is calculated in two different ways: as one triangle, and as the sum of the areas of three small triangles. Read through the proof. Important keys to understanding this proof are the two area formulations:

$$\text{Area}(\triangle ABC) = \frac{1}{2}bh$$

and

$$\text{Area}(\triangle ABC) = \text{Area}(\triangle ADC) + \text{Area}(\triangle CDB) + \text{Area}(\triangle BDA).$$



Here's the proof, using the diagrams above:

$$\begin{aligned} \frac{1}{2}bh &= \text{Area}(\triangle ABC) \\ &= \text{Area}(\triangle ADC) + \text{Area}(\triangle CDB) + \text{Area}(\triangle BDA) \\ &= \frac{1}{2}bx + \frac{1}{2}by + \frac{1}{2}bz \\ &= \frac{1}{2}b(x + y + z). \end{aligned}$$

To divide both sides of an equation by b , we must be sure that $b \neq 0$. If b were zero, we wouldn't have a triangle.

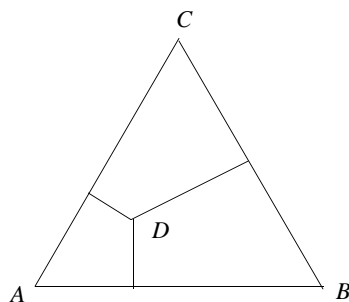
Since $\frac{1}{2}bh = \frac{1}{2}b(x + y + z)$, $b \neq 0$, then $h = x + y + z$.

6. Referring to the material above as little as possible, rewrite the preceding proof in your own words.

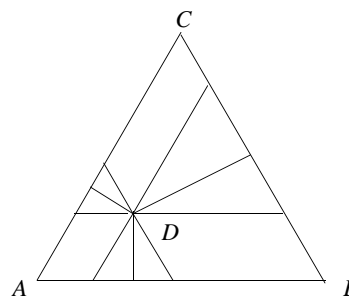
There are other ways to prove Rich's conjecture. "Proofs without words" are designed to prove something or convince you of something using only pictures.

A popular section of *Mathematics Magazine* is called "Proof Without Words."

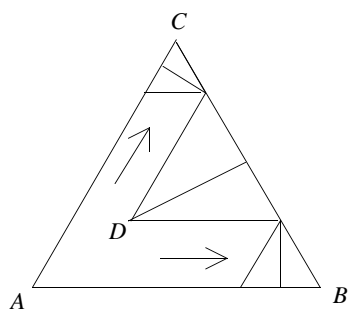
Here's a proof without words of the conjecture which appeared on the cover of the *Mathematics Teacher* (Vol. 85, No. 4, April 1992). Study the pictures and try to make sense of what's going on from one step to the next. After you've looked at the pictures, try to write the words that justify each step.



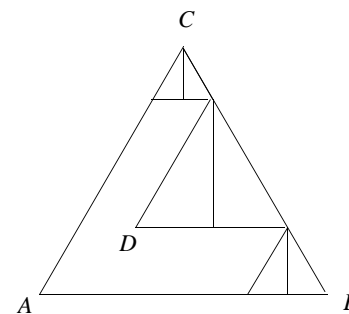
Step 1



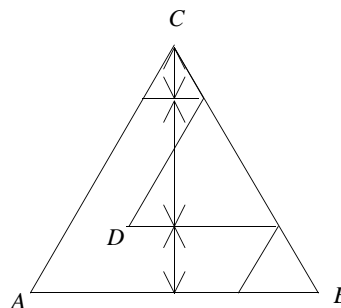
Step 2



Step 3



Step 4



Step 5

For *Step 1* you might write: “We’re given an equilateral triangle with point *D* in the interior. The perpendiculars have been drawn from *D* in order to mark the distances from *D* to the sides of the triangle.”

7. For each step starting with Step 2, write down exactly what changed between that step and the previous one. Make sure to supply reasons that explain why each transformation is valid. (For these pictures to be accepted as a proof, each step must make sense as a true statement. The steps should follow in logical sequence, and should be based on previous steps or known mathematical facts. No magical, mysterious, or trick moves are acceptable.)

Now, rather than just a conjecture, we finally have a theorem:

THEOREM 6.3 *Rich’s Theorem*

The sum of the distances from any point inside an equilateral triangle to the sides of the triangle is equal to the height of the triangle.

One way to describe the situation we have here is to say that it is *stable* or *invariant* on the interior of the triangle. Another less fancy way to describe it is to say that the *function* (the process that starts with a point inside the triangle and then calculates the sum of the distances to the sides of the triangle) takes on *only one value*. What is that one value that the function produces inside the triangle? The function here takes point *D*, which is roving inside the triangle, and assigns to it the sum of the distances from *D* to the sides of the triangle.

Functions that assume only one value no matter what you give them as input are called *constant functions*. So, here’s a mathematical way to state the theorem:

THEOREM 6.3A *Rich’s Theorem Restated*

The function that measures the sum of the distances to the sides of an equilateral triangle is constant for any point in the interior of the triangle.

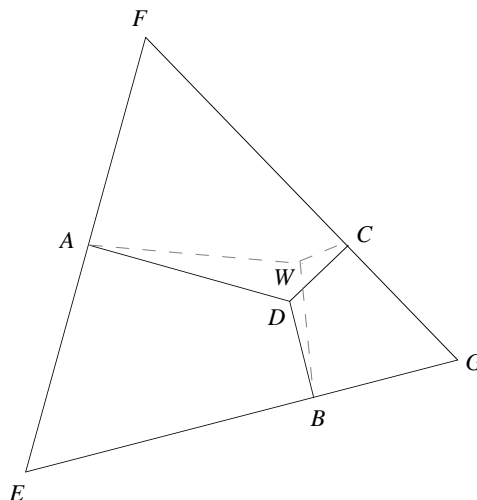
On the other hand, the *burning tent function* is a process that takes a point along the river (a line) and calculates the sum of the distances from you to the point and from that point to the tent. Unlike Rich’s function, the values for the burning tent function *vary*.

This theorem can be used to establish inequalities. The next problem gives an example.

8. In the picture below, $\triangle EFG$ is equilateral, and perpendiculars from D to the sides of the triangle are drawn, going to points A , B , and C . W is some other point inside the triangle. Show that

$$WA + WB + WC > DA + DB + DC.$$

(Hint: Draw perpendiculars to the sides from W , too.)

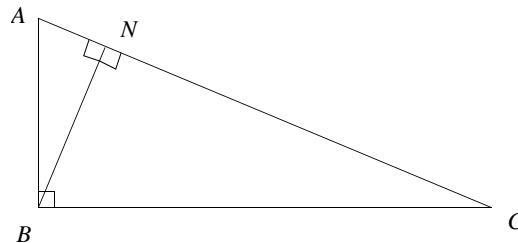


CHECKPOINT.....

9. Given an equilateral triangle of sidelength 8 cm, and a point G inside the triangle, what is the sum of the distances from G to the sides of the triangle?
10. Define each of these words:
- translation
 - rotation
 - function
 - conjecture
 - theorem

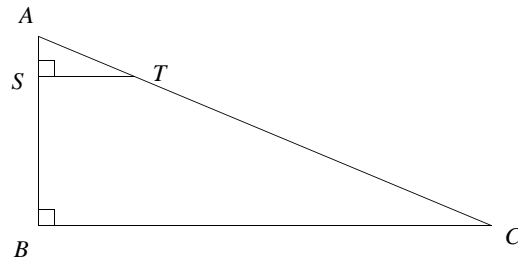
TAKE IT FURTHER.....

11. Use the strategy of calculating area in more than one way to find the length of \overline{BN} if $AB = 5$ and $BC = 12$.



Do you know another way
to find the length of \overline{ST} ?

12. In the triangle below, $AB = 5$, $BC = 12$, and $SB = 4$. Find the length of \overline{ST} by calculating area in more than one way. (Recall that the area of a trapezoid is half the product of the height and the sum of the lengths of the bases.)

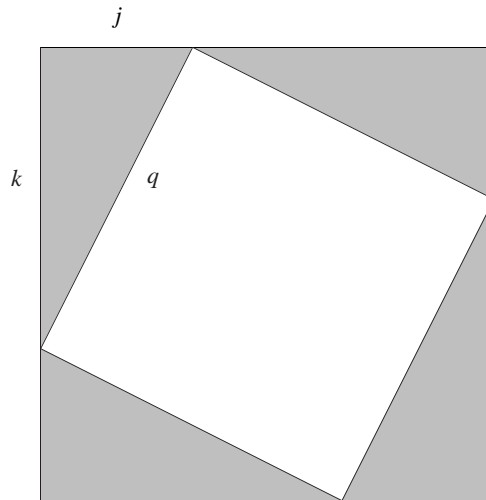


13. Draw a “proof without words” to show that the sum of the lengths of the perpendiculars to the sides of an equilateral triangle from a point anywhere on the triangle is equal to the height of the triangle. (Do not trivialize the problem by using one of the vertices.)

Note that q^2 is represented by the area of the (nonshaded) internal square.

See the *Connected Geometry* module *The Cutting Edge* for more wordless proofs of the Pythagorean Theorem.

- 14. Challenge** Transform the following picture in the manner of a “proof without words” to prove the Pythagorean Theorem. That is, draw another frame like the one below of sidelength $j + k$, and rearrange the shapes inside to show that $j^2 + k^2 = q^2$. Write a sentence or two to explain why you are allowed to make the transformations you did, and why that proves that $j^2 + k^2 = q^2$.



All four shaded triangles are congruent.

VARIATIONS ON A PROBLEM

What would a contour plot for *this* function look like?

Learning to ask yourself “What if things were a bit different?” helps you to break free from narrow ways of thinking, get out of ruts, and solve problems.

The problem in Investigation 6.7 was about the sum of the distances from a point inside an equilateral triangle to the sides of the triangle. The major result was that this sum was constant (and equal to the height of the triangle). You looked at it geometrically, and you also thought about it as a function that assigns a number to a point. The major result says that the function assigns the *same* number to every point inside the triangle.

One often gets insight into a problem by looking closely at the problem’s components (its elements and assumptions) and also by investigating other closely-related variations on the problem, those ones that make slightly different assumptions. So what *were* the components of Rich’s problem?

Rich’s problem included the following five components:

- a. the sum of distances
 - b. from a point
 - c. inside
 - d. an equilateral triangle
 - e. to the sides of the triangle.
1. Suggest one or two reasonable alternatives for each of the five components above. For example, you might replace “sum of distances” with “longest distance.”
 2. Select one of your alternatives, and say what you can about how it might alter the problem. Make a conjecture about how the major result might be affected.

WHERE TO PROCEED FROM HERE

The obvious next step is to investigate one or more of the problems you’ve just posed. There are at least two ways to proceed: working independently (generally with colleagues like your teacher and one or more classmates) or working with the guidance provided in this module.

In the remainder of this investigation, we will explore two variations on the original problem: What happens to the values of the function on the *exterior* of the equilateral triangle? What happens when the triangle is not equilateral?

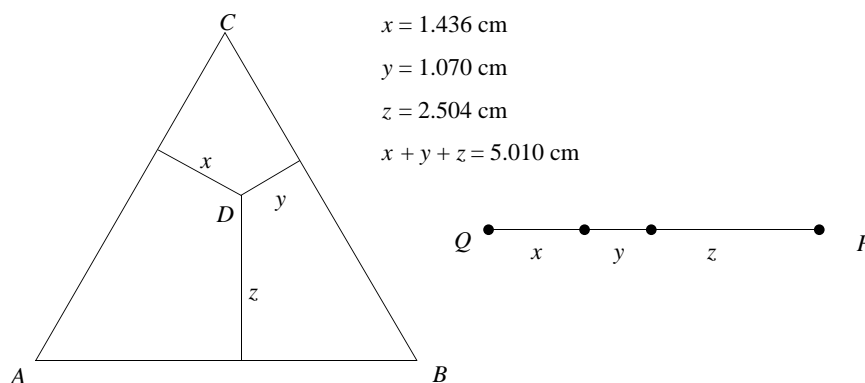
The third section of this module, “The Airport Problem,” guides an extensive investigation of a function that measures distances to the vertices (rather than sides) of a

triangle. A great deal of mathematics has grown up around this particular problem and its implications; related research continues today.

MOVE OUTSIDE THE TRIANGLE

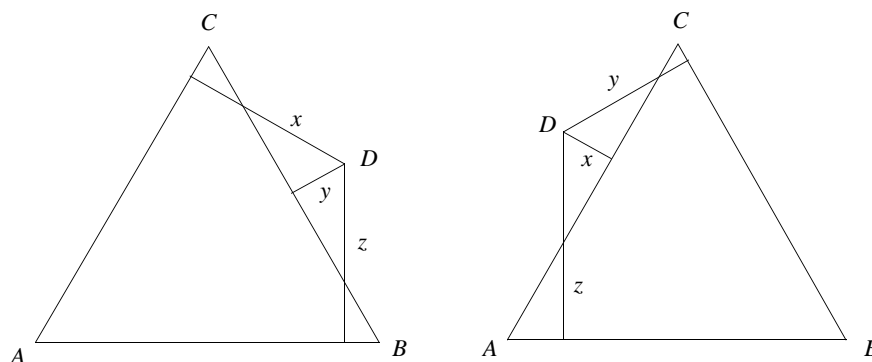
Earlier, when using dynagraphs made with geometry software to explore the function for the sum of the distances to the sides of an equilateral triangle (let's call the function R), did you ever let point D stray outside of the triangle? Make a dynagraph or use your previous construction to try that.

3. Describe what happens when the point strays outside the triangle.

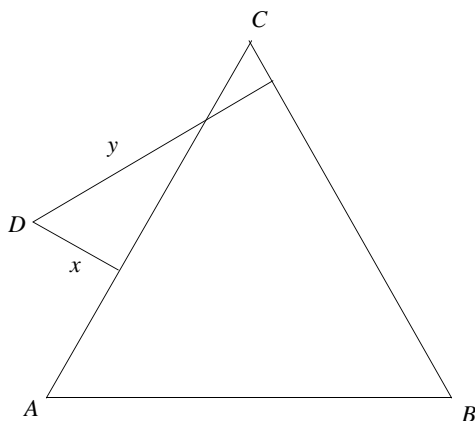


What's the height of this equilateral triangle?

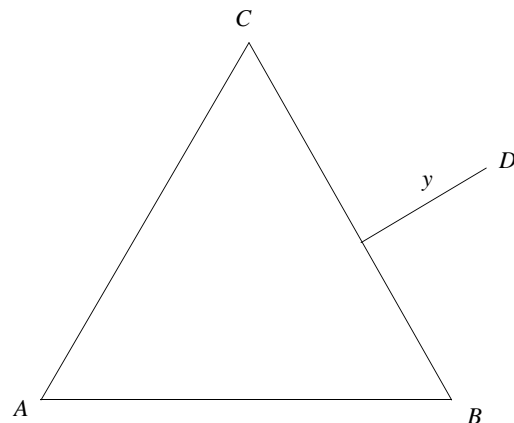
If your dynagraph works like ours, strange things start to happen when you move D outside the triangle. For some D 's, things look fine:



But for other points, D loses a “connecting arm”:

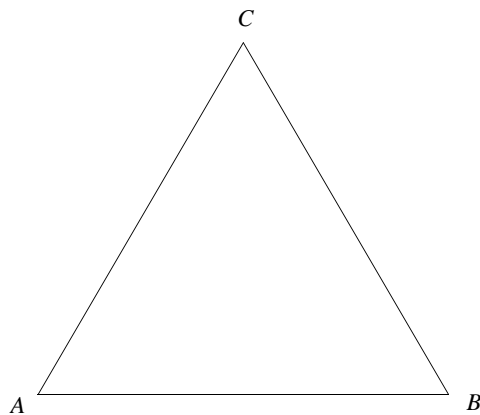


One arm gone ...



or two ...

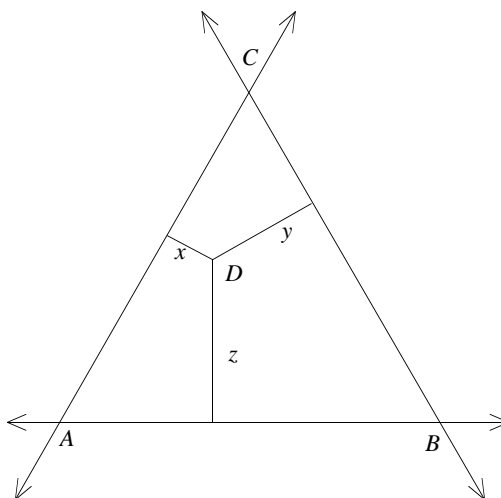
$D \bullet$



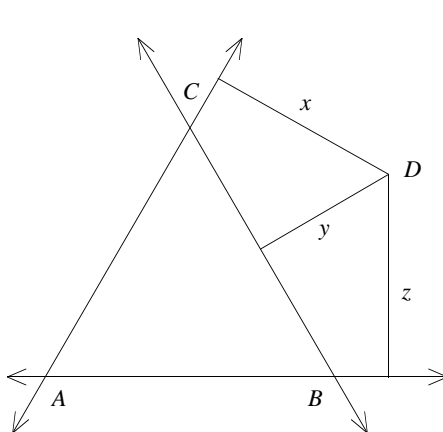
... or even three.

What's wrong here? Why does D sometimes lose some or all of its arms outside the triangle? How does the software you are using handle the calculation of the sum of the distances when D loses an arm? Is what it does reasonable? Is there a reasonable alternative?

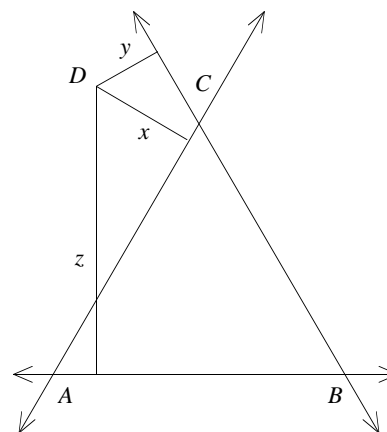
One way to prevent D from losing arms is to think of the sides of the triangles as lines instead of line segments:



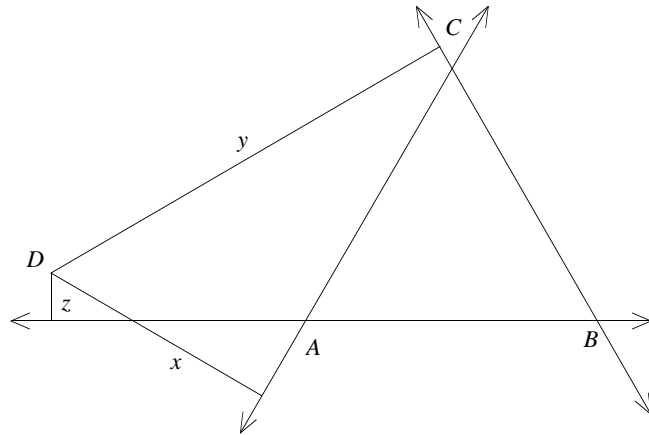
That way you can put D anywhere you want and D will have three arms. You can put it



here ...



or here ...



... or even here.

4. If D lost arms in your sketch, build a new improved sketch by changing the line segments to lines.
5. Experiment with the dynagraph for your new sketch, a generalized version of the function R for the sum of the distances to the sides of the equilateral triangle, and investigate the function's behavior when D moves outside the triangle.
 - a. Is the function constant on the outside?
 - b. Are there any places outside the triangle that produce a value smaller than the value on the interior of the triangle? Is there a minimum value for the function?
 - c. Is there a maximum value for the function if you let D roam over the entire plane? Why?
 - d. Put D somewhere that produces a "nice" value for the function (say, a whole number like 10). Find and draw what you believe to be the contour line for the function at the value you just chose.
6. Make a conjecture about the behavior of the function R outside the triangle. What evidence do you have to support your conjecture?

CONTOUR PLOTS AND SURFACE PLOTS

Do you remember the contour plots and surface plots from the problems in Investigation 6.4?

7. Describe the connection between a contour plot and a surface plot. If you have a contour plot, how can you visualize the corresponding surface plot? If you have a surface plot, how can you visualize the contour plot?
8. Imagine or draw a picture of a dartboard or archery target with a bull's-eye in the middle. If that were a contour plot, what would the corresponding surface plot look like?
9. In Problem 5, you drew what you believed to be a contour line for the function R . Draw three or four more of these contour lines so that you have a simple contour plot for function R . Describe this contour plot as if you were writing to a friend who cannot see it.

You may want to use graph paper.

TAKE IT FURTHER.....

10. Prove that the function R is indeed constant along any one of the contour lines you drew.
11. In writing and with a sketch, describe the surface plot (the three-dimensional model) for the function R . Explain the shape of the surface plot. How might you use the surface plot to explain to a friend that the function is constant on the interior of the equilateral triangle?

WHAT IF THE TRIANGLE ISN'T EQUILATERAL?

Theorem 6.3 (Rich's Theorem) says that from any point inside an equilateral triangle, the sum of the distances to the sides of the triangle is always equal to the height of the triangle. How can this theorem be generalized to include any triangle?

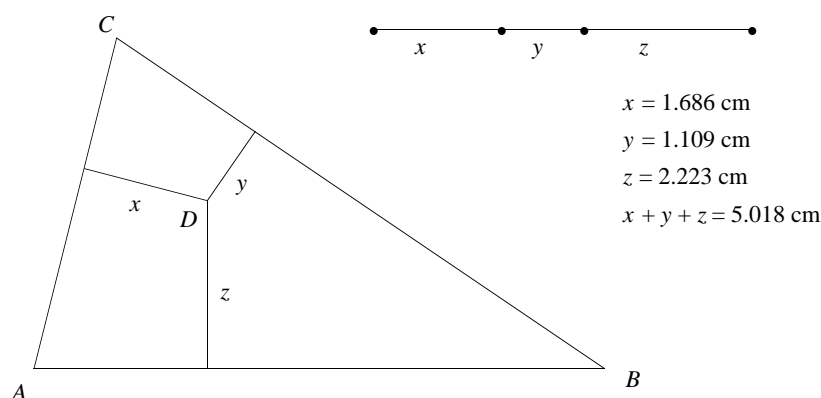
12. How can you decide, by looking at the wording alone (and with no other experiments) whether "the sum of the distances to the sides of a *nonequilateral* triangle will always be equal to the height of the triangle?"

There is a way that you can figure this out without any measuring at all. Think of Rich's original method.

What *are* the important special cases for this problem?

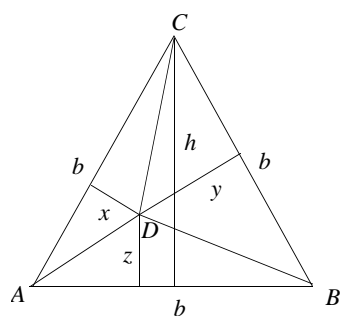
Of course, your measurements will probably be different from the ones shown here.

13. If the triangle isn't equilateral, will the sum of the distances to the sides still be constant?
14. Often, you can obtain a general feeling for a system by investigating its behavior in special cases. Make a dynagraph or a paper drawing to experiment with the function for the sum of the distances to the sides of a nonequilateral triangle.



- a. List some important special cases to be sure to investigate.
- b. Check the special cases, including special points within the special cases.
- c. Write a paragraph or two describing what you learn about the behavior of this function.

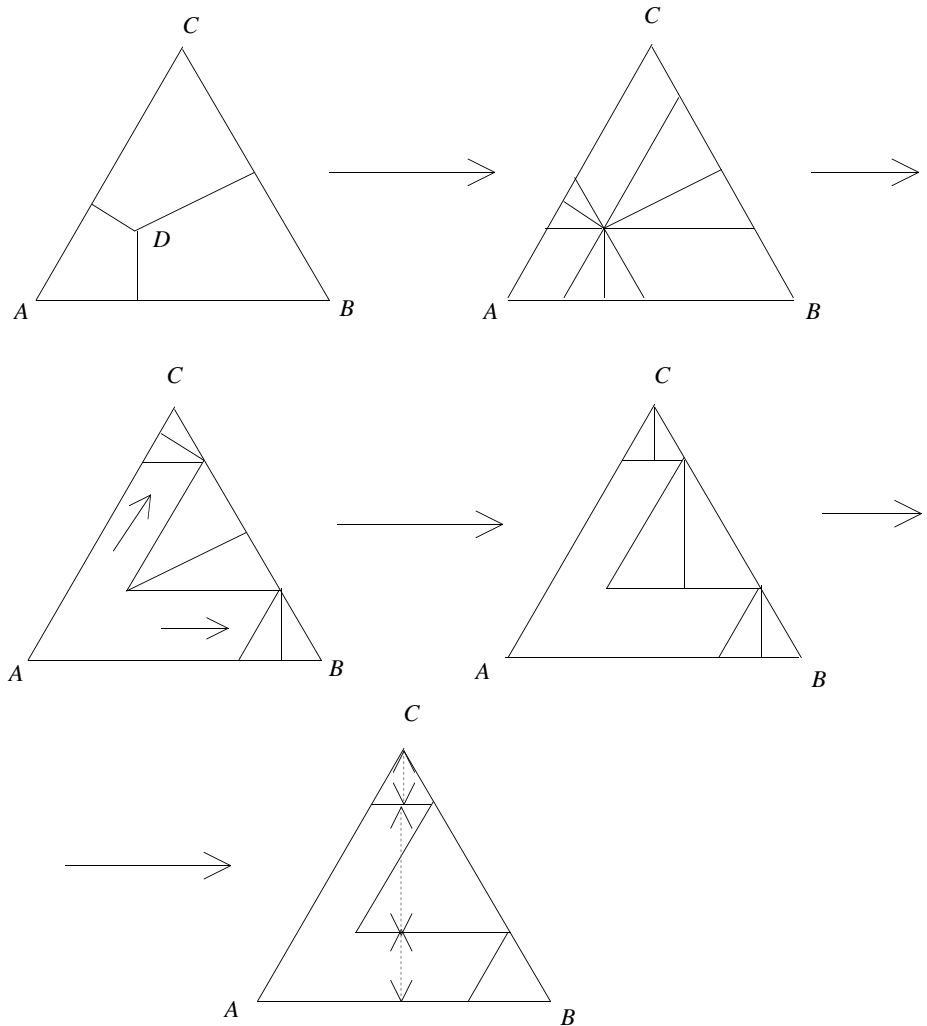
In Investigation 6.7, you looked at two proofs that the sum of the distances to the sides of an equilateral triangle is equal to the height of the triangle. One proof used algebra and calculated the area in two different ways:



$$\begin{aligned}
 \frac{1}{2}bh &= \text{Area}(\triangle ABC) \\
 &= \text{Area}(\triangle ADC) + \text{Area}(\triangle CDB) + \text{Area}(\triangle BDA) \\
 &= \frac{1}{2}bx + \frac{1}{2}by + \frac{1}{2}bz \\
 &= \frac{1}{2}b(x + y + z).
 \end{aligned}$$

Thus, $h = x + y + z$.

The other proof used geometry and the transformations of smaller equilateral triangles and their altitudes:



- 15. Write and Reflect** What goes wrong with each of these proofs when you no longer assume the triangle is equilateral?

What you've done in Problem 15 is another use of proof. By studying what goes wrong with a proof when you change what is given, you can often find new results. You can develop a conjecture for this problem by experimenting, and that's a good thing to do. But by looking at the proofs and what goes wrong with them, you may actually see a way to prove your conjecture.

So, in the case of function R , if the triangle isn't equilateral, the function is no longer constant on the triangle's interior. We'll have to call the function by some other name, say, S (for the Sum of the distances to the sides of a nonequilateral triangle).

16. a. Where is this new function, S , smallest?
- b. Where is it largest?

Here is a better, more precise way to ask these questions:

PROBLEM

How do you know there is even *one* point inside or on the triangle at which the sum of the distances is as small as possible?

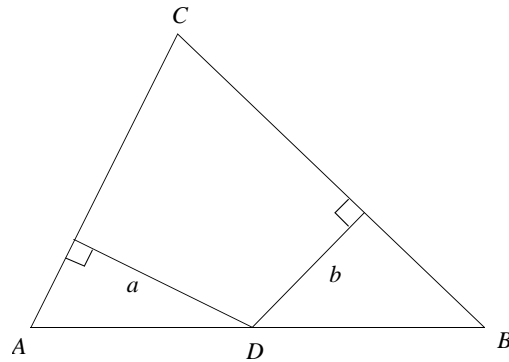
Why aren't these the same questions as those immediately above?

Learning to solve problems often involves learning to ask yourself the right questions, questions that will help pinpoint the most important parts of the problem.

At what point(s) D (inside or on the triangle) is the sum of the distances to the sides of the triangle as small as possible? As large as possible?

17. One more thing: What is that minimum *value* for the function S on the *interior* of the triangle? What is the maximum value? (Does the interior include the triangle itself?)
18. In Problem 16, you looked for the minimum value and the maximum value for the function on the interior of a nonequilateral triangle. In what kinds of triangles are these two values equal? For these triangles, what does that say about the function?
19. Here's a different function:

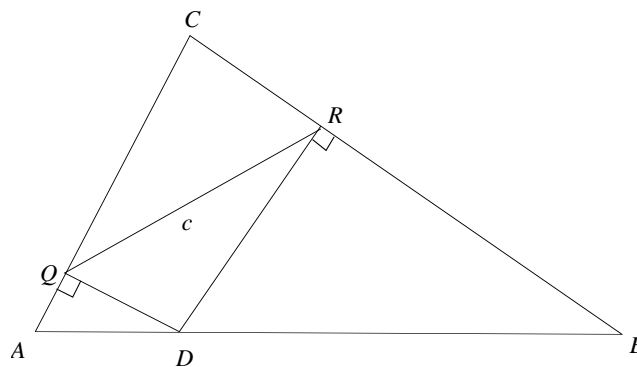
D is a point moving only along \overline{AB} of $\triangle ABC$. The perpendiculars are drawn so that, at each position, the function calculates the sum of the distances to the other two sides of the triangle. What position for D minimizes the sum? What is the minimum *value* for this function? What position for D maximizes the sum, and what is that maximum value? For what kinds of triangles is the function constant along \overline{AB} ? Prove or support all your conjectures.



20. Here's yet another function:

Have you seen this problem before?

Again D moves only along \overline{AB} of $\triangle ABC$, and perpendiculars are drawn to the other sides to calculate the distance, but this time the value of the function is given not by the sum of the distances but by c , the length of \overline{QR} .



- a. What position for D minimizes the value of this function, and what is that minimum value?
- b. What position for D maximizes the value of this function, and what is its maximum value?

CONTOUR AND SURFACE PLOTS FOR THE NONEQUILATERAL CASE

You probably know the question that's coming next: What does a contour plot and a surface plot look like for the function S (that is, when the defining triangle isn't equilateral)?

Your method should not rely on already having a surface plot.

- 21.** Invent a method that *you* could use to find the contour lines for this function over a nonequilateral triangle. Use that method on different triangles to investigate the contour plots for a scalene triangle, an isosceles triangle, and a right triangle.
- 22.** Sketch a surface plot for the function over a nonequilateral triangle. Try jotting down a few notes to yourself first in order to keep all your information straight. Which three points might be the easiest ones to start with to get yourself going?

MAKING CONNECTIONS IN MATHEMATICS

PERSPECTIVE ON DR. FAN CHUNG

At the end of this essay there's a problem to solve about the sum of the distances to the *vertices* of a triangle. The problem was a very important one to Bell Labs.

The original interview appeared in *Math Horizons*, September 1995. © The Mathematical Association of America. All rights reserved.

Dr. Fan Chung is a mathematician who worked for 20 years at AT&T Bell Laboratories (Bell Labs) and Bellcore; she works now as a professor of mathematics at the University of Pennsylvania. In one interview she spoke about the excitement and power of making connections among various branches of mathematics. Previous investigations in this module looked at the sum of the distances from a point to the sides of an equilateral triangle, and two different proofs gave you an opportunity to make connections between geometric and algebraic solutions to the same problem. Just thinking about the problem in terms of a *function* that assigns a number to a point on the plane makes a strong connection between geometry and other mathematics.

The following remarks by Dr. Chung are adapted from an interview with her by Don Albers, editor of *Math Horizons*, a journal of mathematics intended for undergraduate students, but accessible to high school students interested in mathematics. Its purpose is “to expand both the career and intellectual horizons of students.” *Math Horizons* is published four times during the academic year in Washington, D.C., by the Mathematical Association of America. The MAA is devoted mainly to the interests of college mathematics students, but there are MAA student chapters which work with high school students to sponsor student workshops, mathematics career conferences, conferences in which students present their own papers, and many other student activities.

“Don’t be intimidated!” is Fan Chung’s advice to young women considering careers in mathematics. “I have seen many people get discouraged because they see mathematics as full of deep incomprehensible theories. Yet there is no reason to feel that way. In mathematics, whatever you learn is yours and you build it up—one step at a time. It’s not like a real-time game of winning and losing. You win if you benefit from the power, rigor, and beauty of mathematics. It is a *big* win if you discover a new principle or solve a tough problem.”

Chung regards herself as luckier than many mathematicians. “As an undergraduate in Taiwan, I was surrounded by good friends and many women mathematicians. We enjoyed talking about mathematics and helping each other. A large part of education is learning from your peers, not just the professors. Seeing other women perform well is a great confidence builder, too.” By that logic, Fan has built up the confidence of many other women, for she has performed extremely well as a student and researcher. She enjoyed twenty years of successful and productive work at Bell Labs and Bellcore; she has since returned to academic life as a professor of mathematics.

Combinatorics is the branch of mathematics that deals with counting techniques. It is sometimes called “counting without counting.” For example, combinatorics can tell you how many different poker hands can be dealt from a standard 52-card deck.

Go is a two-person board game that originated in China. In this game, one player captures territory from the other by surrounding the other player’s pieces. It is generally considered to be more complex than chess. Hex might be thought of as a kind of extended tic-tac-toe on a board that is a tiling of hexagons. Players alternately claim the spaces on the board, trying both to block the other player’s chain and to build their own chain of pieces all the way across the board.

Chung recalls having fun with combinatorics during high school and only later discovering connections to other branches of mathematics, in addition to many applications. She is convinced that students profit enormously from establishing connections to other branches of mathematics, saying mathematics can be “like playing a game of Go or Hex. If your territory is all connected together, then every piece is strong and useful. On the other hand, if the parts are separated, then they are weak and not effective.” She aims to make students aware of the power of connections and applications. “If you learn lots of theorems without actually using them, it is like a rich man who never spends his money. There is no difference between that and having no money at all.” She argues that such experiences will make students more employable—as better researchers, better teachers and learners, and better problem solvers.

Her own appreciation for making connections in mathematics dramatically increased when she started working in information technology in 1974. With her new doctorate in hand, she joined the technical staff of Bell Labs, which was richly populated with research mathematicians and scientists, including Nobel Prize winners. She remembers being intimidated by “all of those great name tags” during her first days on the job. “But I got over it, and very soon I discovered that, if you just put your hands out in the hallway, you’d catch a problem.”

How did Fan Chung “catch problems in the hallway?” She says that the hardest part was establishing communication. “You need to have a willingness to find out what problems others are working on. Finding the right problem is often the main part of the work in establishing the connection. Frequently, a good problem from someone else will give you a push in the right direction and the next thing you know you have another good problem. You make mathematical friends and share the fun!” Over the past 20 years, she has been remarkably successful working with others. Nearly half of her 180 papers have been done in collaboration.

Collaborating with others to produce a paper often means making many revisions to reflect the ideas of everyone involved. Fan Chung believes such care and attention is very important. When she arrived at Bell Labs and wrote her first paper under the supervision of Henry Pollak, she realized “it was absolutely clear that he was reading my paper with care. I really appreciate what he’s done for my papers and for the example he set.”

Many years later, when Chung reminded Pollak that her first paper was rewritten eight times, Pollak replied that his first paper had been revised 24 times. It’s no surprise Chung believes that both faculty and students would profit greatly from increased attention to writing and reading.

Discrete mathematics deals with phenomena that *do not* change continuously—for example, the number of diagonals in a polygon is a function of the number of sides, but that function changes in integral jumps (which are discrete) rather than changing continuously.

What do you think Fan Chung means by “the binary universe?”

Chung has continued to make connections with new mathematicians and with new mathematics. Along with the growth in technology during recent decades, there also has been significant growth in mathematics, particularly in discrete mathematics. “Mathematics is more important than ever in dealing with all the hard problems arising from the advances of technology,” says Chung. “The interaction of combinatorics and other areas of mathematics opens up many exciting directions. It is like opening an old treasure box at the same time that you find modern power tools. So you have precious crystals in one hand, a laser gun in another, and the light can go much further.”

“There are many wonderful ideas from discrete mathematics that students need to know about. Because we live in the information age, many challenging problems arise in our binary universe. It is essential for students to be able to connect the mathematics they learn in the classroom to problems we face in this information age!”

In spite of spending most of her professional life working in an applications environment, Chung says that a large part of her drive to solve problems comes from the beauty of mathematics. She says that “the dividing line in mathematics is not ‘pure’ versus ‘applied’; it is ‘good’ versus ‘bad.’ Bad mathematics is cooked up artificially and will vanish from view very quickly.” Good mathematics, according to Chung, is characterized by its impact and longevity.

A PROBLEM FOR BELL LABS

The problem below, one that Bell Labs had to solve, is very much like some of the shortest path problems you have seen in earlier parts of this module.

Telephone companies have traditionally provided a special rate for corporations that make a large number of phone calls between branch offices located across the country. When this system was first introduced, the phone company provided what it called “dedicated telephone lines” and set up the system so that the rate it charged a client was based on the minimum amount of wire needed to connect the various offices. The amount of wire needed to connect two offices was easy to figure out, but when there were three or more offices involved, it got a little more complicated.

Find a map that will tell you the mileage between these cities.

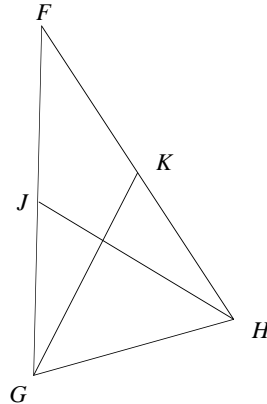
Here's an example to think about involving three cities:

1. Suppose a company had offices in New York City; Miami, Florida; and Kansas City, Missouri.
 - a. Would you run telephone lines from each city to both of the others?
 - b. Would you run lines from New York to Miami and then to Kansas City, or from New York to Kansas City and then to Miami, or from Miami to New York and then to Kansas City?
 - c. How many miles of telephone line do you think it would take to connect all three offices?

One client found a way to make the “minimum distance” (amount of wire needed) between its three offices even smaller by setting up a (false) fourth office.

2. **Write and Reflect** Approximate and draw the location of the three cities above, or set up the situation on a computer and experiment with some locations for that fourth office. Can you find a location through which the three telephone lines could be routed? The sum of those *three* distances should be less than the minimum sum of the *two* distances that could be used to directly connect the three offices (*without* using the fourth office).
3. **Write and Reflect** If so, is this location inside or outside the triangle formed by the three cities? Do you think there's a spot that minimizes that distance? Explain.

1. Triangle FGH is isosceles, with $GF = HF$. \overline{GK} and \overline{HJ} are medians.



Choose the correct symbol to compare the relative sizes or values of each pair of objects named. You may use $=$, $>$, $<$, or ng (not enough information given).

- a. GJ _____ FJ
 - b. GJ _____ GH
 - c. FK _____ FJ
 - d. $FJ + JH$ _____ $FK + KH$
 - e. $\text{Area}(\triangle GJH)$ _____ $\text{Area}(\triangle HKG)$
 - f. $\angle KHG$ _____ $\angle JGK$
 - g. $\text{Area}(\triangle FKG)$ _____ $\text{Area}(\triangle HKG)$
 - h. Distance from G to K _____ Distance from G to \overline{FH}
 - i. JK _____ GH
2. What if $\triangle FGH$ were scalene and \overline{GK} and \overline{HJ} were still medians? Which of the relationships in Problem 1 would stay the same and which would change? For example, $GJ = FJ$ in Problem 1a. Is this still true if $\triangle FGH$ is not isosceles? For each relationship in Problem 1, write “same” if the relationship stays the same, or choose the correct new symbol if the relationship changes. Try to do this exercise without drawing; instead try to visualize the scalene triangle. When you have finished, you can draw by hand or use geometry software to construct a scalene triangle to see if you are right.

3. Define each of the following terms.

equilateral triangle

isosceles triangle

scalene triangle

obtuse triangle

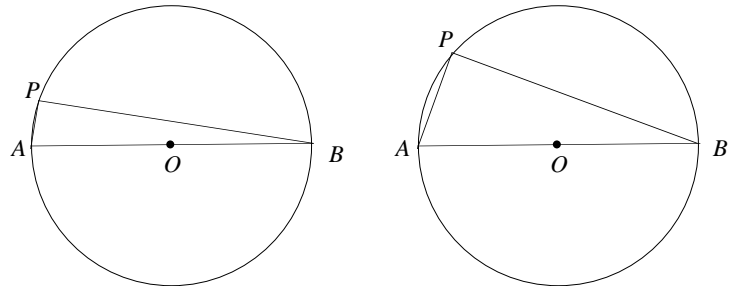
acute triangle

parallel

angle bisector

altitude of a triangle

4. **Write and Reflect** Make a list of everything you know that is true about equilateral triangles. Once you have a complete list, decide which characteristics are true for all triangles; which are true for some, but not all, other triangles; and which are true only for equilateral triangles. Find a way to present this information so that other students can read and understand it easily. The format is up to you.
5. The term *function* has appeared many times in this module (“the function R ,” “constant function,” “a function that calculates the sum of the distances . . .”). Explain what a function is. Since this term has not been officially defined for you in this module, write what you understand it to mean, give some new examples if you can, and write any confusions or questions that you have about the term.
6. Pictured on the next page is a circle with center O , diameter \overline{AB} , a point P on the circle, chords \overline{AP} and \overline{PB} , and $AO = 2$ cm. Assume that P can move around the circle, but that the circle itself is a fixed size. As P moves to each new location on the circle, some things change and some are invariant. For example, as P moves around the circle, the length of \overline{AP} changes, at times increasing in length, and at times decreasing. As P moves around the circle clockwise from A to B , which measures change and which are constant?



- a. For each of the measures listed below, indicate whether the measure is constant or changing.
- b. For each of the measures in part a that are constant, state what that constant value is or how it could be found.
- c. For each of the measures in part a that are changing, explain how they change—use a graph or a clear written explanation.

Measures:

- i. $m\angle APB$
- ii. the distance from P to O
- iii. the perimeter of $\triangle APB$
- iv. the area of $\triangle APB$
- v. the ratio of the circumference of the circle to the diameter of the circle
- vi. the sum of the distances $AP + PB$
- vii. the ratio of AP to BA
- viii. the ratio of AP to PB
- ix. the distance from M to N , where M is the midpoint of \overline{AP} and N is the midpoint of \overline{PB}

GETTING STARTED

Structure your group like you think such a consulting group might be structured.

Why bother with a theory that ignores such realities as topography, politics, and land use? In fact, such a theory is essential. One *must* have a way of saying, “if all else were equal, the best spot would be” Then, and only then, can one have a standard from which to evaluate plans that (almost inevitably) depart from the model.

Three neighboring cities, all about the same size, decide to share the cost of building a new airport. They hire *your group* as consultants to find possible locations for the airport.

1. On a map of your state, find three cities of about the same size. Make *this* your airport project. In your group, discuss where you think the airport should go and why. Then write a short preliminary report describing your conclusions.

In the following investigations, you will develop the mathematics to determine some theoretically-best spots for the airport (ignoring things like lakes, mountains, waste dumps, and so on). After you’ve developed your conjectures and proved your theorems (there’ll be plenty), you’ll write a final report to the city councils explaining how you’ve chosen the airport location and why it’s best. When you write that final report, you should show where some *theoretically*-best spots are located and also include practical concerns or reasons (if any) for *not* recommending these mathematically-determined spots.

WHAT IS MEANT BY BEST?

In locating the best spot for the airport, you have to determine what the people in the cities care about. For example, they might want the location of the airport to be “fair” in the sense that each city should be approximately the same distance from the airport.

FOR DISCUSSION

Come up with some other possible meanings or definitions for “best.” Discuss the advantages and disadvantages of each definition.

1. Which of the definitions of “best” would you recommend? Why? Are any of them the same? Which would be acceptable to a city council?
2. Explain how to locate the best spot for the airport if the cities agree to the “fairness solution” suggested above.
3. Give an example of locations for three cities where the “fair” location for the airport seems just plain silly.

THE ENVIRONMENTAL SOLUTION

Why only “if the cities make roughly equal use ...”?

The environment is probably best served by minimizing the total distance traveled by all users. If the cities make roughly equal use of the airport, this environmental solution is also the most economical, as it minimizes the amount of roadway that must be constructed. Not only can roadbuilding costs be substantial at the outset, but maintenance costs can be high since the highway department charges for every mile of roadway that must be maintained. The shorter the roads, the lower the total cost for building and operating the airport.

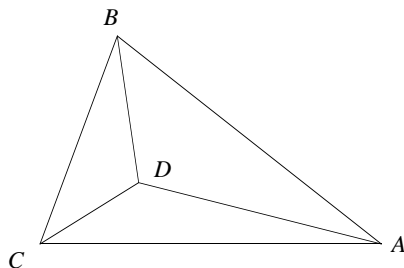
4. How is the “environmental solution” different from the “fairness solution?” Would they ever be the same? If so, under what conditions?

This environmentally- and economically-best spot turns out to be more difficult to locate than the “fair” spot, but it is far more practical. To locate it, we also must develop and use some beautiful mathematics. So, we have another *optimization* problem,

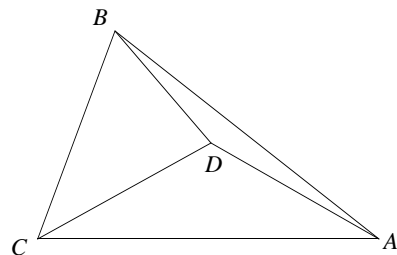
the airport problem. We want to minimize the sum of the distances from the airport to the three cities. If we label the cities as vertices of a triangle, then our problem can be stated the following way:

PROBLEM *The Airport Problem*

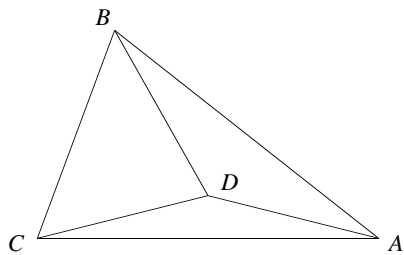
Where should we put point D if we want to make $DA + DB + DC$ as small as possible?



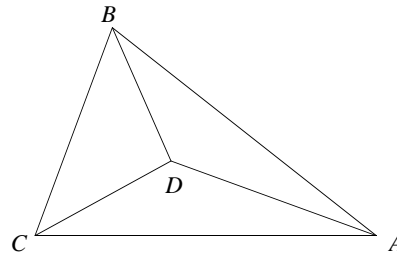
Should it go here?



Or here?

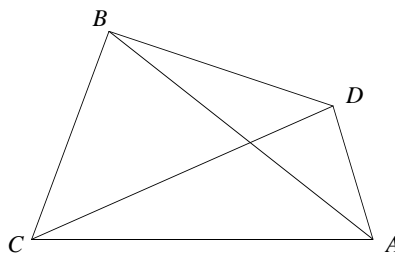


Or here?



Or here?

No ... definitely not *there*.

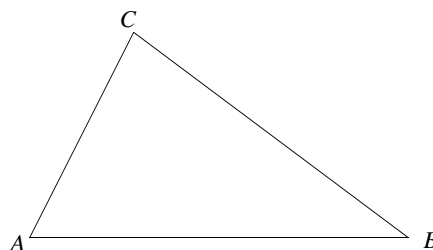


Or maybe even here?

The airport problem can be thought of as a system that has three fixed points (the three cities) and a moving point (the variable site for the future airport). When you think about it this way, you imagine D , the variable, moving around the plane. At each spot, you check the value of $DA + DB + DC$, trying to find the spot that makes the sum as small as possible.

5. This kind of image—here the image of a continuously-varying D coupled to a varying “sum of distances”—comes up again and again in mathematics. Give an example of another problem that could be viewed as a continuous system.
6. Sometimes you can get a sense of a system by investigating its behavior in special cases. Draw $\triangle ABC$ on a piece of paper so that $AB = 9$ (inches, say), $AC = 6$, and $BC = 8$. Place D at several points and calculate the sum of the distances to the vertices. Of all the possible positions for D that you try, which one makes $DA + DB + DC$ smallest?

For example, what is $DA + DB + DC$ if D is at A ? At B ? At C ? What if D is at the midpoint of \overline{BC} or at the foot of the perpendicular from C to \overline{AB} ? What if D is at the intersection of the two lines you just drew (from A to the midpoint of \overline{BC} , and the perpendicular from C)?



A 6–8–9 triangle

7. **Write and Reflect** What do you think? Develop a conjecture about where to put the airport to minimize the total distance to three cities (for *your* three cities or for *any* three cities). Justify your conjecture with an argument.

Finding the point inside a triangle that minimizes the sum of the distances to the vertices is not an easy problem. Just coming up with a reasonable conjecture may take some time. In addition to working on a conjecture by yourself for awhile (always a good way to start), you might try looking through some books to see if you can find any ideas that might apply. You also might exchange your ideas with other people to test or fine tune them, or to get new ideas.

The next investigation suggests three different ways of thinking about this difficult problem: analyzing special cases, using mechanical models, and using geometry software to experiment with the problem.

If you already have some of your own ideas about how to solve the problem, you may not need to look at these sections.

So, if you get stuck and are looking for new ideas, browse through one or more of the sections in Investigation 6.13. The object is for you to find your own way to come up with a reasonable conjecture about where to put the airport.

A LONG-RANGE PROBLEM

The *real problem*, though, is not finding the conjecture. You might well be able to look that up somewhere. The *real problem* is

Did you ever see an invention, understand completely how it worked, but wonder how anyone ever thought of it in the first place?

8. Present a plausible explanation of how you or any other reasonable person, doing reasonably-expectable things, might come up with the conjecture on your own.

This is a problem that you cannot complete until you've finished some other work (like coming up with the conjecture). We place it here to alert you, in advance, that part of your task is to *keep track of your thinking*. Later, after you've had a chance to investigate the airport problem, you will come back to this question (about the process of coming up with the conjecture) and write about what led you to your ideas.

SPECIAL CASES AND MODELS

Here are three ideas for directions you might take in working on the airport problem:
1) explore special cases, 2) make a mechanical model, and 3) make a computer simulation.

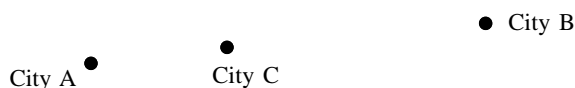
IDEA 1: LOOK AT SIMPLER PROBLEMS AND SPECIAL CASES

Your results may surprise some people. Talk it over in your group until you are all prepared to show anybody in class the best spot or spots for the airport in this situation.

1. What if there are only two cities? Where could the airport go?



2. What if there are three cities, but they do *not* describe a triangle? (That is, what if they are collinear?) Make an argument for an airport site that minimizes the total distance. Can you find a “fair” solution (a spot that is equally distant to each city)?



Will the best spot move very far if you move one of the cities only a little bit?

3. What if the cities are *almost* collinear? Make a first guess based on the results from Problem 2, and find the total distance for your guess. Then check some other spots that are nearby (and a couple of farther ones) to see if your first guess seems to be the best one.



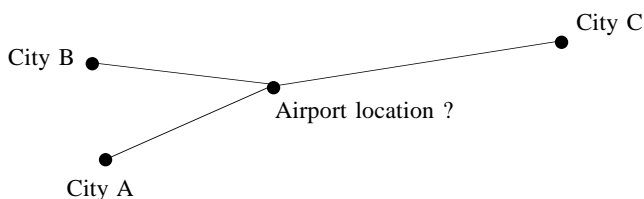
Small diagrams make it difficult to measure the effects of different choices for locating the airport. We recommend using a whole sheet of paper and measuring carefully. If you're using technology, then use the entire screen for your diagram.

What is it that makes this a simpler case? Are there other triangular "special cases" that you'd expect to yield insights?

In this case, the surprise seems to be telling us that *distance* is not the thing that matters.

Looking for special and extreme cases is a very useful way to think about problems. Putting the cities in a line was a special case that made it easier to find a best location. Now try some special kinds of triangles in the hope that the special cases will help you make good guesses about the general case. Here are two ways that might simplify the three-city problem.

4. What if one city is very far away from the other two? The sketch below doesn't necessarily show the best position for the airport. Where (roughly) would that position be? Explain your reasoning. How would the best position change if City C were twice as far away from \overline{AB} ? This may be a good question to investigate using geometry software.

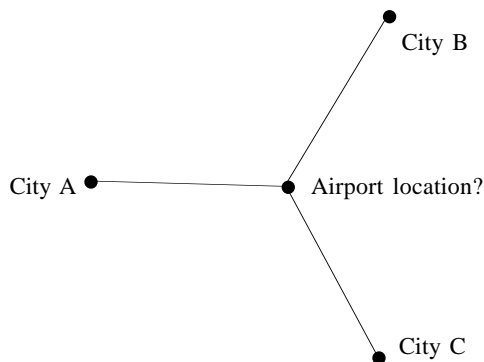


This result seems surprising—shouldn't the airport location be pulled along when you move City C? Surprises are often good sources of clues.

Also, in most triangles, the airport ends up in the "middle" of the cities. Is there a special-case triangle that would put the airport in a precise "middle"? Special-case triangles might include ones that are short and fat, long and skinny, and maybe right, equilateral, or isosceles triangles.

What does an equilateral triangle have going for it in this case that other triangles don't? What exactly is meant by the "center" of an equilateral triangle? Do other triangles have centers?

5. What if the cities form an equilateral triangle? The center of the triangle looks like a good spot for the airport. Test enough locations to see if the center spot really minimizes the total distance when the cities are equidistant from each other.



Here the cities form an equilateral triangle.

Now consider the general problem—for *any* triangular arrangement of the three cities.

This may require you to measure other things like angles, distances from the sides of the triangle made by the cities and so on.

Describing your method is one thing; convincing others that it really *does* locate the best point is another. That's where some more mathematics comes in.

6. a. Find a way to locate the spot for the airport that is a minimum total distance from the three cities. Check to see if your method works for isosceles, obtuse, or other special triangles. (If it works for most but not all triangles, that's OK. Just describe what's different about the cases where it fails.)
- b. Explain *why* your method works, and include examples. Be prepared to *convince* people that your method of choosing a spot really does minimize distance.

The guess-and-check method makes some strong assumptions about the nature of a problem:

You can't check *every* point in a region, even in a tiny one. Between any two points, no matter how close, there is always another point. Perhaps, tucked in somewhere among the very worst of locations will be one that is far better than the best spot you've found in a good-looking region! If small changes in the location of the airport *could* make such a big difference, then the process of checking your best guess with some nearby spots wouldn't be reliable.

Some situations *are* that messy. Is *this* one of them?

7. Experiment enough to get a sense for how the total distance changes with location. Give an argument to support this claim:

“If I move the airport just a little bit, the sum of the distances to the cities won’t change much.”

Or, stated more formally:

“The sum of the distances varies continuously with the location of the airport.”

Maybe a contour plot would help here.

8. Could there be more than one spot where the sum of the distances to the vertices is as small as possible? Defend your answer. After all, there were *many* best spots in the two-city airport problem.

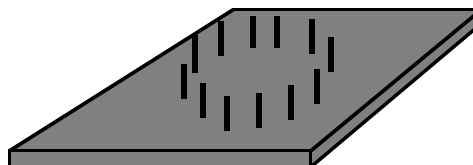
IDEA 2: LOOK AT A MECHANICAL MODEL

The more ways you have for thinking about a problem, the more ways of checking whether a method gives reasonable results. When people are quick to recognize reasonableness, we often say they have “common sense.” Common sense comes from experience and from thinking about things in flexible and various ways.

Sometimes a machine or physical model can help you experiment with a system. Three physical models for investigating the airport problem are described in this section of the investigation. If you already have a conjecture, a model can help you test it. If you don’t have a conjecture, one of these three models may help you come up with one.

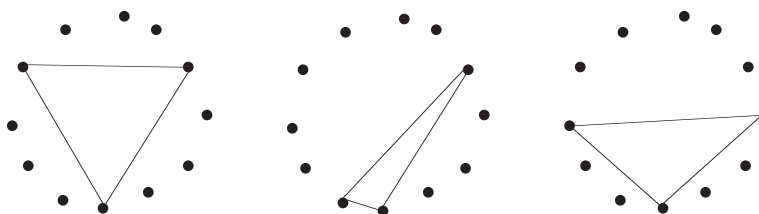
MODEL 1: STRING AND NAILS

Take a flat piece of wood and pound nails into it, more or less in a circle.



You could also use three push pins in wood or cardboard, but it isn't quite as strong. Rearrange the push pins to represent different configurations of the cities.

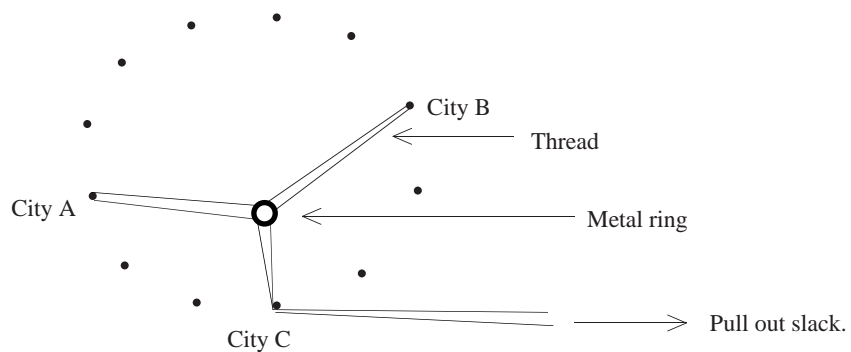
With enough nails in your circle, you can choose three to represent cities in a triangle of almost any shape.



Pick three nails that give you the shape you want.

A model can be crude and still help you think about a problem. As your ideas develop, you can improve the model if you want.

The model is completed with a small metal ring to represent the airport and a strong, slippery thread or string (nylon works well) whose length represents the sum of the distances from the airport to the cities (actually twice the sum, because the thread must be doubled to loop around the nails). The following sketch shows how the thread is passed through the ring and is looped around the nails.



9. Explain how this device can be used to locate the best airport location for three cities. Why does it work?
10. Use the model for several arrangements of three cities. Record each experiment by placing a piece of paper inside the circle of nails and tracing out the roads to the best airport for each arrangement.
11. For a given set of three cities, could there be more than one spot where the sum of the distances to the cities is as small as possible? Defend your answer.

.....
WAYS TO THINK ABOUT IT

The thing we want to minimize is the sum of the distances to the cities. This is a *number*, and you can calculate it for *any* point.

So, we have a new function. Let's call it the *airport function* and name it a . We'll call the variable point U . The cities are labeled A , B , and C . For any point U ,

$$a(U) = UA + UB + UC.$$

The notation $a(U)$ is read " a of U " or " a evaluated at U ." In other words, $a(U)$ is the output of the function a when U is the input.

We can now restate the airport problem:

The Airport Problem: What point U produces a minimum value for the airport function a ?

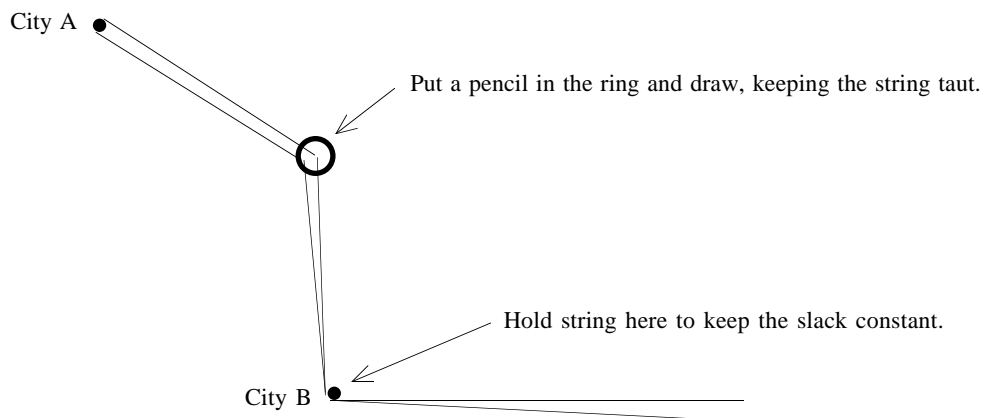
.....

12. Use the string and nails device to make a contour plot for the airport function a . Use your contour plot to defend your answer to Problem 11.
13. Explain how the contour plot could be used to solve the problem of locating an airport for three cities where an obstacle (such as a power plant) prevents using the best spot.

This technique might be useful for the cities you chose in Problem 1 of Investigation 6.11.

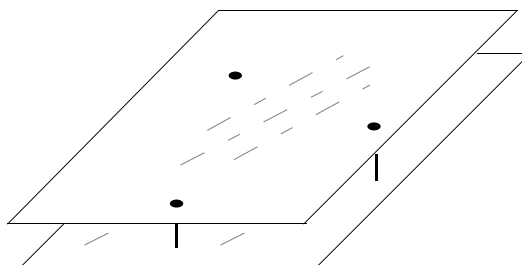
A big part of doing mathematics is learning to ask the right kinds of questions.

14. Experiment with the two-city airport problem using this model. What do the contour lines look like this time?



MODEL 2: SOAP FILMS

Clamp together two pieces of clear plexiglass, and drill three holes through them. Remove the clamps, and install “spacers” in the holes (use nuts and bolts) so that the plates are at least $\frac{1}{4}$ inch apart and parallel to each other.



Dip the frame in a soap solution. The network of soap film will settle into a favored position.

15. a. What might make it reasonable to expect that this position is the solution to the airport problem? That is, why should soap films produce the same kind of minimum distance that solves the airport problem?

- b. Try a similar soap-film technique with four posts arranged in a quadrilateral. Most likely, the result will not look like a single-point location that solves an “airport problem.” What kind of problem *is* being solved, or modeled, by the soap film?

MODEL 3: STRING AND WEIGHTS

Do NOT drill holes in your desk!

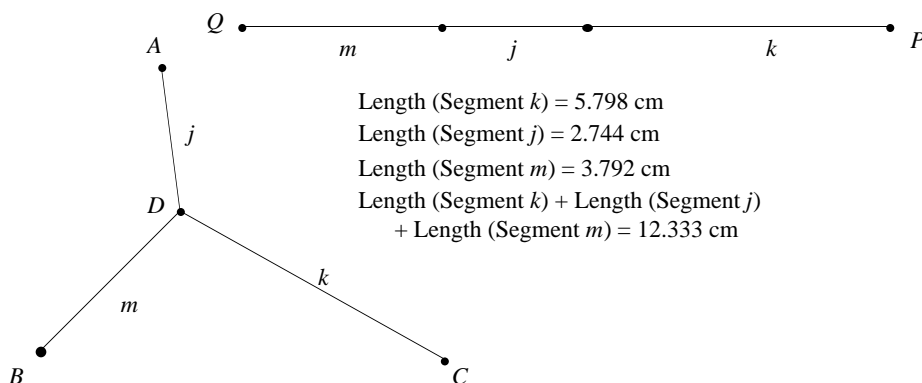
Drill three holes in a board, arranging the holes where your cities would be. Take three strings, each about a yard long, and tie equal weights to one end of each. Pull each string up through a hole in the board; each weight should hang below the board. Tie the three free ends together, then raise the board enough to let the weights dangle freely. Adjust the knot until the system feels in equilibrium.

16. a. Clearly, any physical system does *something*. Why is it reasonable to expect that the equilibrium point for this system solves the airport problem?
- b. How does the behavior of a four-hole system of strings and weights differ from that of a four-post system of soap films? What problem does the four-hole system of strings and weights seem to solve?

IDEA 3: USE A COMPUTER

You can also use geometry software to model the airport problem.

Build a dynagraph that lets you see the value of the airport function both numerically and geometrically. Here is such a dynagraph:



As you move D around, the total distance increases and decreases. The length QP is the value of the airport function for the point D . Because you are trying to minimize this length, have the software trace the locus of P . As you move D , what will happen to P when the total distance is minimized?

Work patiently and plot a lot of points.

- 17.** Use your geometry software to draw the contour lines for the airport function.
- 18. Write and Reflect** State any conjectures you have so far for the airport problem. Provide evidence for your conjectures.
- 19.** You can also use your computer to keep track of angles in the airport problem. Try it; the pattern(s) you find here might support a conjecture or two.

TESTING THE CONJECTURE

The experiments in the previous sections make it seem reasonable to state the following conjecture:

CONJECTURE *The Airport Conjecture*

What does “too big?” mean? If it means “bigger than 60° ,” then the conjecture is true but of very limited value.

If three cities are arranged in a triangle, and no angle of that triangle is too big, then the best place for the airport is the spot where the roads to the cities form 120° angles with each other.

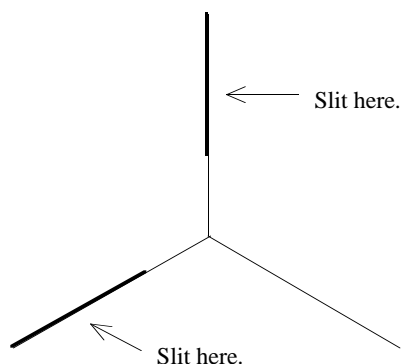
1. Write and Reflect Take time *now* to return to Problem 8 in Investigation 6.12:

Present a plausible explanation of how you or any other reasonable person, doing reasonably expectable things might come up with the conjecture on your own.

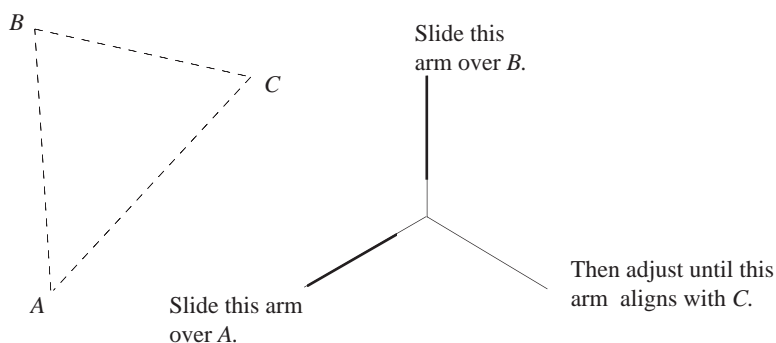
120-DEGREE GADGETS

Before attempting to prove the conjecture, it seems reasonable to *test* the conjecture. Jim Sandefur, a mathematician from Georgetown University, proposed the following device for locating the 120° spot and thus testing the conjecture.

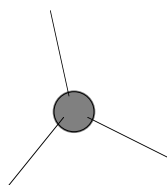
Represent the three cities with three pins stuck in a shoe box and label them *A*, *B*, and *C*. Draw three 120° angles about a point on an overhead transparency, and then make slits in two of the “arms” of your drawing—from the ends to a spot close to the common vertex.



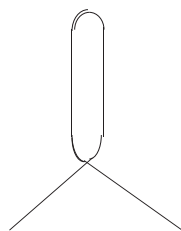
To find the 120° spot, slide one slit over the pin at A , the other one over the pin at B , and then “wiggle” the transparency until the third arm lines up with C .



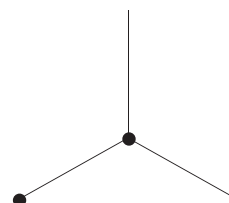
- Construct a “ 120° gadget” that has three legs radiating out from a center at 120° from each other. It could be toothpicks in a ball of clay, a bent paper clip, a shoebox device, a drawing on a transparency, or a computer construction that can be moved about the screen.



toothpicks in clay



bent paper clip



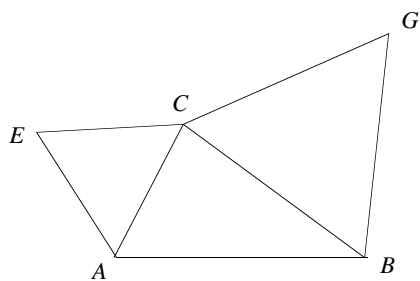
construction with geometry software

"Does it always ...?" "Does it ever ...?" Questions like these require careful thinking. Clearly, you cannot check all possible configurations. What experiments might lead you to make reasonable conjectures about these questions? How might you support (not necessarily prove) your conjectures?

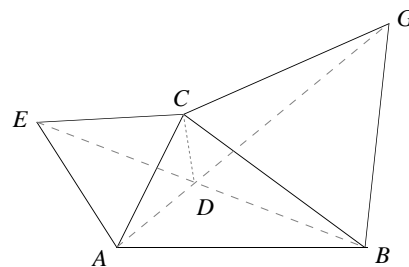
3. Try your gadget.

- Does the gadget always find an airport site?** That is, is it *always* possible to adjust your 120° gadget on a diagram of three cities until each leg passes through one city, regardless of the configuration of cities?
- Does the gadget ever find more than one airport?** That is, is there ever a configuration of cities for which there might be *more* than one location for the center of your gadget—the intersection of its three legs—from which each leg passes through one city?
- Does the gadget find the same airport site(s) that you would find with your other methods?** Does a site found with the gadget lie where your other methods tell you it should?

4. One of our students proposed a construction to locate the 120° spot. The construction takes two steps: (1) build equilateral triangles on two sides of the original triangle; (2) connect the new vertices of the equilateral triangles to the opposite vertices of the original triangle. According to the student, the intersection D of those two connecting lines is the 120° spot, that is, $\angle ADB$, $\angle ADC$, and $\angle CDB$ are all 120° .



Step 1



Step 2

Does this method work? Why?

5. Things look good for the 120° conjecture, but why does it fail for triangles with wide angles? Is there a 120° spot in all triangles? When the 120° spot doesn't work, what is the spot that minimizes the airport function?

CHECKPOINT.....

6. Define each of the following words.

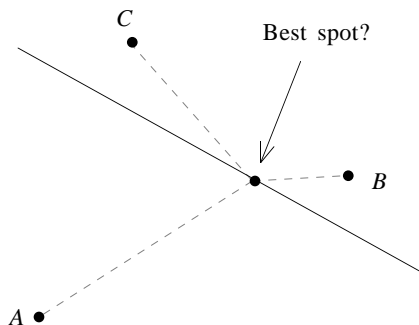
contour line
 contour plot
 function
 equilateral triangle
 conjecture
 collinear

What does *algorithm* mean? There's quite a history behind the word. It evolved from the public name, al-Khwarismi, of a mathematician whose real name was Jafar Mohammed ibn Musa.

7. Give an algorithm for locating the “fair solution” (where the airport would be equally distant from each city) in any triangle.

TAKE IT FURTHER.....

8. Explain how the contour lines for the airport function might be thought of as generalized ellipses. How would you define an “ellipse” with *four* foci? How could you tie some string to draw one?
9. Suppose the airport for three cities had to be placed on a specific line (there's an existing highway that must be close to the airport). Now the problem is to locate, among the points *on the line*, the one that minimizes the airport function. Propose and explain a method for finding that spot.



10. What does the airport function's surface plot look like?

ESTABLISHING THE CONJECTURE

PERSPECTIVE ON FERMAT

In this essay, you will learn about the long history of the airport problem and about the life and work of Fermat. Do you think he called it the “airport problem”?

For the last 300 years, dozens of mathematicians have looked at the airport problem and have attempted to glean new insights from it. Fermat, Steiner, Gauss, Toricelli, Viviani, and Fesbender are just a few of the people who have played important roles in the history of the problem. Along with many others, these mathematicians have produced results that are both beautiful and practical.

The first recorded version of the airport problem occurs in the writing of Pierre de Fermat. In an essay on optimization, he appears to have posed the problem as an application of his work. He worded it in this way: “Let [they] who do not approve of my method attempt the solution of the following problem: Given three points in the plane, find a fourth point such that the sum of its distances to the three given points is a minimum!”¹

Fermat is one of the most famous names in mathematics, but he never considered his relationship to mathematics as anything more than a hobby. He never published any of his work, and he never wrote any long treatises on his mathematical ideas. Yet he was clearly one of the most original thinkers in mathematics. He made major breakthroughs in number theory and contributed to the beginnings of analytic geometry, calculus, and probability theory. Fermat was born in 1601 in Beaumont de Lomagne, France. He followed the family profession of law, eventually becoming a King’s councilor in the local parliament at Toulouse. His mathematical work consists almost entirely of correspondence with other mathematicians of his day and of notes written in the margins of the many mathematical books he read for his consuming hobby.

In 1641, Fermat scribbled one of his cryptic messages in the margin of a copy of a text on number theory. The message was that he had proved that there are no positive integers x , y , and z that satisfy the equation $x^n + y^n = z^n$ if n is a whole number

¹Kuhn, H. W., “Steiner’s Problem Revisited,” *Studies in Optimization*, Dantzig, G. B. and B. C. Eaves, eds: Washington D.C., Mathematical Association of America, 1993.

If $n = 2$, the equation becomes $x^2 + y^2 = z^2$, which should remind you of the Pythagorean Theorem. Any set of three integers that satisfy this equation can be used as sidelengths of a right triangle. For this reason, a set of such integers is called a *Pythagorean triple*. Pythagorean triples and ways to generate them are discussed in the *Connected Geometry* module *The Cutting Edge*.

The PBS series *Nova* presented a fascinating program about Wiles and his work called “The Proof” (Program #2414, broadcast October 28, 1997). A complete transcript of this program, an interview with Wiles, and related information about Fermat’s Last Theorem can be found on the Internet at www.pbs.org/wgbh/nova/proof.

greater than 2. This was quite surprising, because if n is 2, there are infinitely many triples x , y , and z that work (such as 3, 4, and 5). Fermat claimed he could prove this conjecture (that no whole number solutions to $x^n + y^n = z^n$ exist if $n > 2$), but that the proof was too long to fit in the margin of his book. For close to 400 years, no one was able to produce a proof, and it’s quite likely that Fermat’s proof (if he had one) was flawed. Although it remained unproved for centuries, Fermat’s conjecture has long been known as “Fermat’s Last Theorem.”

Attempts at finding a proof led to the development of a whole branch of mathematics, algebraic number theory, and occupied the entire careers of dozens of people. A breakthrough came in 1983 from Gerd Faltings, a German mathematician, but it didn’t settle the problem. Faltings’s work implied that for $n > 2$, the equation $x^n + y^n = z^n$ had at most a *finite number* of solutions for any n . All that remained was to show that the finite number was always 0, but that was no small task. On the experimental front, researchers used computers to show that, if there were an exponent n and some integers x , y , and z that satisfied the equation $x^n + y^n = z^n$, then n would have to be bigger than 300,000.

Andrew Wiles, a mathematics professor at Princeton University, read about Fermat’s Last Theorem in a book in his public library in Cambridge, England when he was 10 years old. He was determined to prove the theorem some day, but he knew that he would need to learn a lot more mathematics. At Princeton, he worked for seven years on this problem. On June 23, 1993, back in his hometown of Cambridge, he announced to the world that he has proved Fermat’s Last Theorem! An error was later discovered in his work. After a year of work with the Cambridge mathematician Richard Taylor, Wiles fixed the error and repaired the proof.

The proof that Wiles created is very long (150 pages!) and complex. It is based on very advanced 20th century mathematics. So if Fermat had a valid proof (which most mathematicians, including Wiles, doubt), it must have been very different. The famous conjecture has finally been proved, but we are left with the mystery of whether Fermat really had a proof, and, if so, what it was.

ONE PROOF OF THE AIRPORT CONJECTURE

We can’t prove Fermat’s Last Theorem here, but we can certainly look at a proof of the solution to his “airport” problem.

A proof often requires a new insight, a new strategy, or a new way of looking at things. The process can look quite mysterious from the outside, where the bizarre

turn of thought that made the proof possible seems to come completely out of the blue. In fact, these ideas do not come out of the blue: no one invents a proof out of thin air. People walk into situations with a lifetime of experience and with a collection of habits of mind developed over a lifetime. These experiences and habits include general principles. Here are some examples:

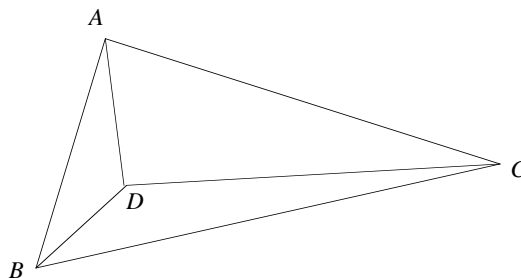
- If a system varies continuously, and at one point its value is 5 and at another point its value is 10, then, somewhere between these two points, its value must be 8.
- The shortest way to get from A to B is to travel along the segment from A to B .

Some people believe it helps to memorize things like this and then apply them according to some problem-solving strategy. The development of good habits of mind doesn't work that way. Habits are things you do routinely. You don't even have to think about them. You develop habits over time. Good habits of mind are essential ingredients, but rarely the only ones. Proofs arise from a combination of *general* habits and *specific* experience with the problem.

Most people improve their ability to create proofs by studying the proofs of others.

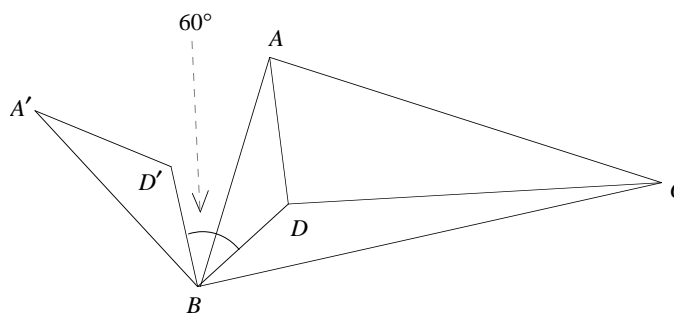
Following someone else's proof to see how it works often provides a good picture of the inner workings of a complex problem. The proof outlined here was invented by the German mathematician J. E. Hoffman in 1929. As you'll see, Hoffman essentially applied the strategy outlined in the "burning tent" problem, but with a few more complications.

Recall the setup: You have a system of three fixed points and a moving point D . You want to minimize the sum $DA + DB + DC$. In the picture below (and all the following ones) D is picked not as the *best* spot, but just *some* spot that will later be improved upon.

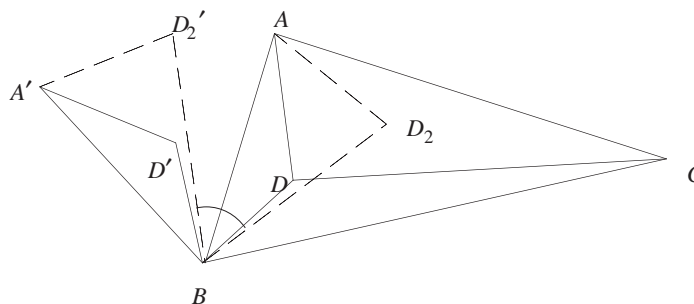


Hoffman must have had the 120° conjecture in mind before doing the rotations. Maybe something special about the 120° point made this construction look reasonable.

Step 1 Hoffman thought of a similar problem that is easier to solve using an equivalent system (a system of three lengths). In this new system, the best spot is simpler to locate. Hoffman's ingenious construction rotates the smallest triangle, BDA , 60° counterclockwise around B . It's not clear how Hoffman thought of this, but it's the trick that makes the proof work.



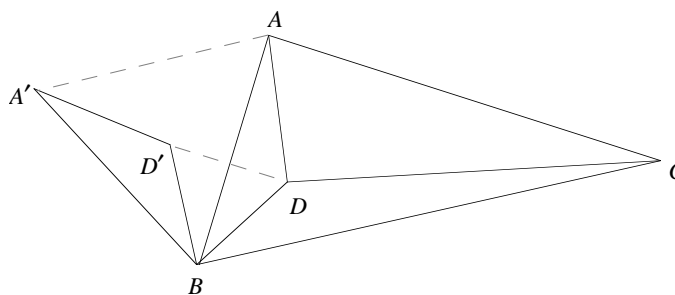
Think carefully about the parts in the construction. A' is the result of rotating A around B through 60° . So, the position of A' is in no way influenced by our arbitrary choice for placing D . If D had originally been placed somewhere else, say at D_2 , its image D_2' after rotation would have been different, but the location of A' would not have been affected.



New $D_1(D_2)$ means new $D'(D_2')$ but the same old A' .

Now look again at the picture above. $\triangle BDD'$ is equilateral because $\overline{D'B}$ is a 60° rotation of \overline{DB} . (An isosceles triangle where the angle between congruent sides

measures 60° is also equilateral.) Below, the equilateral triangle is completed to make it more visible.



Triangles ABA' and DBD' are equilateral.

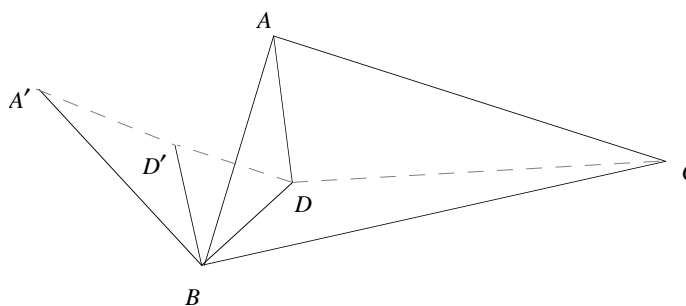
Because $\triangle BDD'$ is equilateral, the distance from D to D' is the same as from D to B . Look at that a couple of times, because that's the fact that makes the proof work: $DD' = DB$.

$\triangle ADB$ is congruent to $\triangle A'D'B$. Rotation doesn't change any of the original triangle's measurements.

The last thing to notice is that $D'A' = DA$ because D' to A' is the rotated version of D to A . So keep track of these two things:

$$DD' = DB \text{ and } D'A' = DA.$$

In the following picture, segments \overline{CD} , $\overline{DD'}$, and $\overline{D'A'}$ are dashed. No matter where we put D , the construction ensures that $DD' = DB$ and $D'A' = DA$.



So what does all this have to do with the distance from the cities to the airport? The sum of the distances from the airport to the cities is $DC + DB + DA$. But that's the same as $CD + DD' + D'A'$:

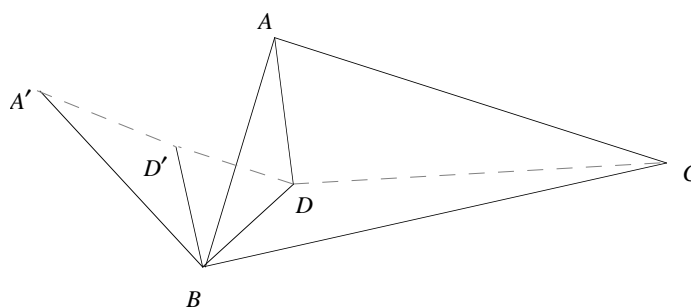
$$DC + DB + DA = CD + DD' + D'A'.$$

That means that the dashed line has the same total length as the sum of the distances to the airport. So finding the *minimum* value for the airport function is the same as finding a place for D where $CD + DD' + D'A'$ is as small as possible.

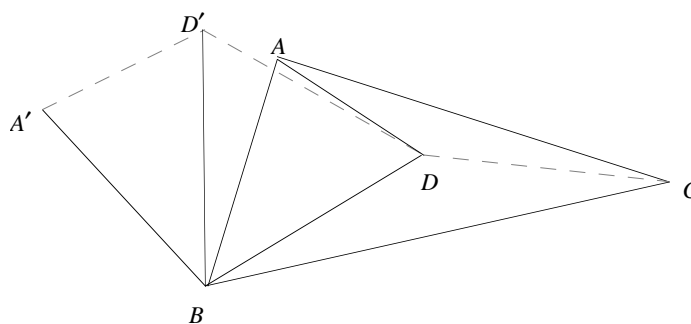
Step 2 A geometry software construction of the above setup helps show what to do next. Draw $\triangle ABC$, create a point D , and rotate $\triangle BDA$ 60° about B . Then mark the segments as shown in the previous picture.

You can now move D about, looking for a place that makes $CD + DD' + D'A'$ as small as possible. But notice that the segments \overline{CD} , $\overline{DD'}$, and $\overline{D'A'}$ form a path from C to A' . We must put D in a place that minimizes the total trip from C to A' .

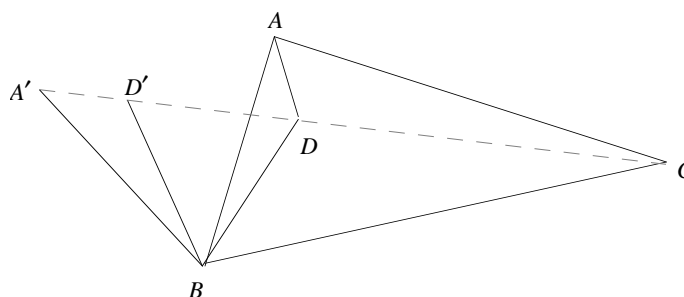
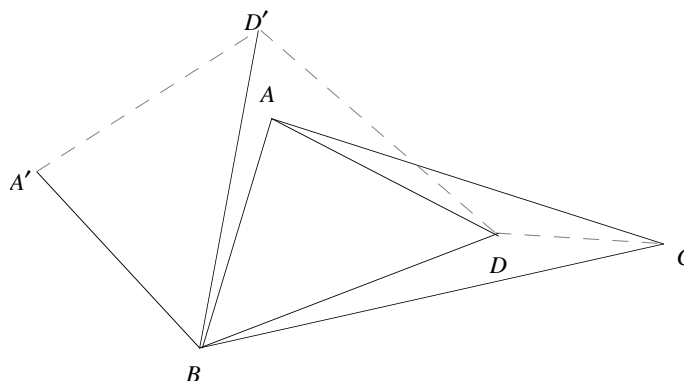
Should I put D here?



... or here?



No ... definitely not there.

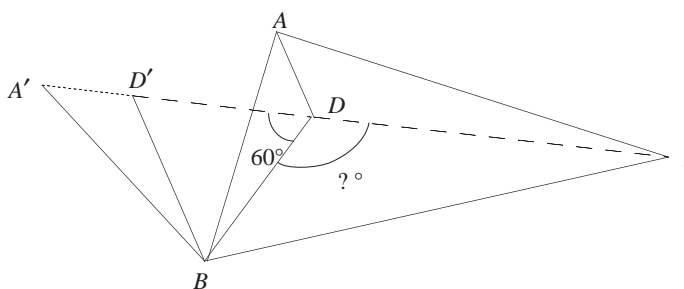


What about here?

Remember, the construction is set up to keep
 $CD + DD' + D'A' =$
 $DC + DB + DA.$
Hoffman was pretty clever!

That'll do it. The shortest path from C to A' is along a straight line, so if we can line things up in such a way that \overline{CD} , $\overline{DD'}$, and $\overline{D'A'}$ all lie along a straight line, then we've minimized the sum $CD + DD' + D'A'$. And that means that $CD + BD + DA$ is also as small as possible.

Now think about the angle made by B , D , and D' . That's inside the little equilateral triangle we made by rotating $\triangle BDA$. If $\triangle BDD'$ is equilateral, $m\angle BDD'$ has to be 60° .

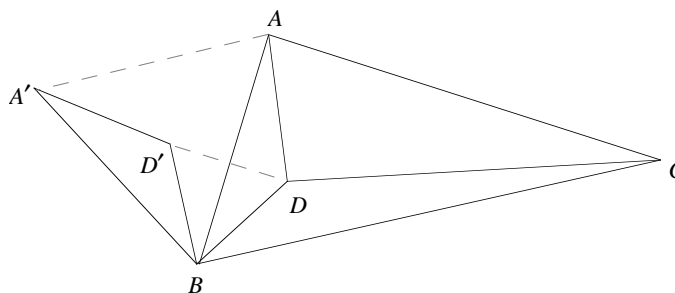


1. If A' , D' , D and C are all collinear, what is the measure of $\angle CDB$?
2. Does this prove what we want to prove? In particular, is the measure of $\angle ADB$ also 120° ?
3. Copy the figure above (build another equilateral triangle using a *different* side), and connect its outer vertex with the opposite vertex of $\triangle ABC$. The proof says that the best location for the airport is somewhere on that connecting line. If the proof is valid, that line will go through the point D already created by the proof. Check it out.

SUMMARY OF THE PROOF

When you see a proof in a mathematics text or in a mathematics paper, usually many details are left out, as are explanations of the steps. In fact, if you saw Hoffman's proof in a journal, you might see something like this:

Suppose D is any point interior to $\triangle ABC$. Rotate $\triangle BDA$ 60° counter-clockwise as in this figure:



It can easily be seen that

$$DA + DB + DC = D'A' + D'D + CD,$$

and the right side of this equation is minimized when A' , D' , D , and C are collinear. This happens precisely when $m\angle BDC$ is 120° . By symmetry, the other angles around D also have measure 120° , establishing the result.

4. Explain this summary of the proof in your own words, filling in all the gaps.

5. What are the advantages and disadvantages of presenting proofs in a way that leaves out many of the details and the explanations for the steps?
6. **Write and Reflect** Prepare a presentation for the class in which you explain Hoffman's proof in your own words.
7. **Write and Reflect** How do you think Hoffman might have thought of the main idea for the proof?

We can't close this investigation without stating the theorem we have just proved. Here's *one* way to say it:

THEOREM 6.4

The point is known as *Fermat's point* in honor of the inventor of the problem. People who pose good problems are often more famous than people who solve them.

The best place to put the airport is where the roads make 120° angles with each other, unless there is no such place inside the triangle.

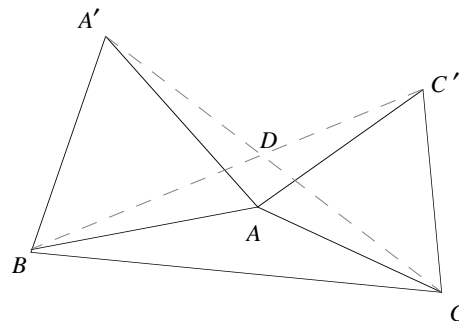
8. Improve the statement of the theorem by putting it into your own words and by making it more precise; that is, tell exactly when there is no such point and what to do in those cases.

USING THE PROOF

Having the proof of something can make you feel good. It can make you think you *really* understand something; it lets you know *why* something is true. But proofs are used in mathematics for more than making people happy. Coming up with a proof can be a useful research technique in the sense that it helps you find new facts and do and learn new things. In the case of the airport problem, Hoffman's work was designed to prove that the best airport location within the triangle will have 120° angles between the roads around it (if there is such a place). But it does more than that. It also provides a method for *finding* that point without hunting around for it.

If you need ideas, look back at Problem 3 in this investigation, and Problem 4 in Investigation 6.14.

9. Invent a construction for the best airport location.
10. Review the construction in Problem 4 of Investigation 6.14. Use the Hoffman proof to explain why this construction produces a point such that all the angles around it have measure 120° .
11. Go back to some of your earlier diagrams, and compare this construction's "best" place to the "best" place you had found earlier.
12. You've performed the Hoffman construction on *two* sides of the original triangle (Problem 3). Predict what would happen if you did the construction for the third side. Give a reason for your prediction. (Then check it by performing the construction.)
13. The proof also explains why the 120° conjecture won't work for triangles with very big angles. The picture below shows what happens when one angle made by the cities is too big.



- a. Why would D be a silly location for the airport in this case?
- b. What *is* the best spot for the airport in this case?
- c. For what kinds of triangles (be precise) will the Fermat point lie outside the triangle made by the location of the three cities?

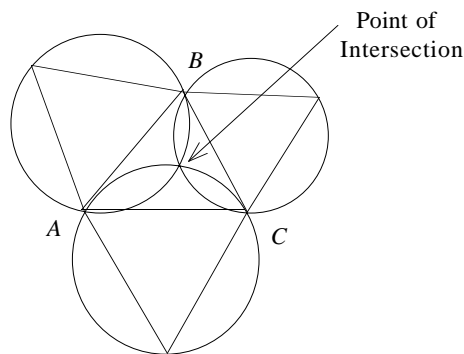
TORICELLI'S CONSTRUCTION

Despite the fact that Toricelli proposed this solution before 1640, it was not published until 1659—by Viviani, one of Toricelli's students.

In 1993, a student of one of the authors discovered an elegant construction. (See Problem 4 in Investigation 6.14.) Compare it to the following, similar construction usually attributed to the Italian mathematician Toricelli, who gave the first known simple geometric solution to the problem.

Step 1 Toricelli constructed equilateral triangles on the outside of each of the sides of the given $\triangle ABC$.

Step 2 Next, Toricelli circumscribed circles around each of these equilateral triangles. The three circles intersect at the desired point.



The book, *Exercitationes Geometricae*, published in 1647, showed that, when you construct lines from any two of the given points to the Fermat point, the angle formed has a measure of 120° . (This is the way we stated the airport theorem in this book.)

The book *Doctrine and Application of Fluxions* was published in 1750. It contained a proof that the three lines joining the outside vertices of the equilateral triangles to the opposite vertices of the given triangle also produces the Fermat point. This is what you proved if you did Problem 4 of Investigation 6.14.

14. Show that Toricelli's circle construction really does produce the Fermat point. That is, show that the circles really meet at *one* point and that point is the 120° spot.
15. **Write and Reflect** How might Toricelli have *come up with* the construction in Steps 1 and 2? That is, what *reasonable* things might have led to this approach?

THE AIRPORT REVISITED

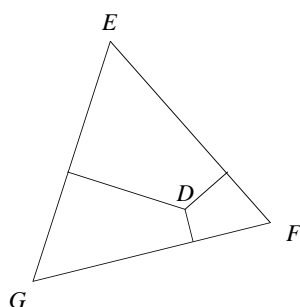
Lemma comes from Greek for *proposition* or *statement*. The German word for theorem is *satz* and the word for lemma is *hilfsatz*, a “helper-theorem.”

One thing leads to another. In mathematics, when one result is used to prove another, the first result may be called a *lemma*. This investigation presents a chain of ideas that uses a theorem we studied earlier (Theorem 6.3 in Investigation 6.7) as a lemma to prove the 120° result for the airport problem. This time, you’ll have to fill in a few links in the chain.

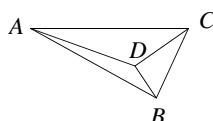
But why *another* proof? Hasn’t Hoffman’s proof (see Investigation 6.15) already elevated the airport conjecture into the Airport Theorem? Isn’t it convincing enough?

Yes and no. Yes, the Airport Theorem is established. But a proof is more than a convincing argument. In fact, many theorems are proved long after people are convinced of them (Fermat’s Last Theorem is an excellent example). One of the most important and useful functions of proof in mathematics is that proofs establish logical *connections* between results. These connections can be used to gain insights into established facts and to invent new conjectures and facts.

It works the other way, too. Looking for logical connections among established results is a good habit of mind for finding new proofs. So, pretend you haven’t seen a proof and that we’re back in the situation where we have a conjecture that the Fermat point is the best place to put the airport. Let’s see if we can find a new reason to believe this.



... to the sides of an equilateral triangle.

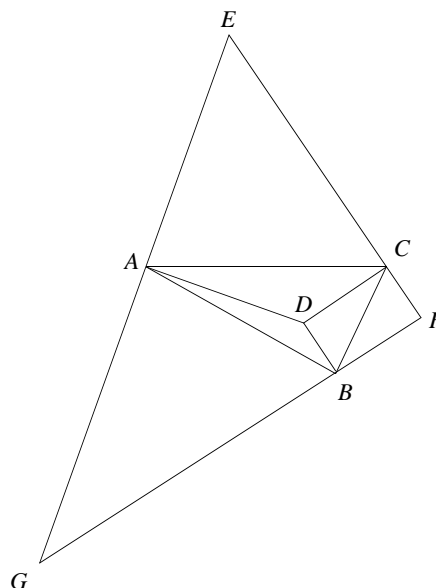


... to the vertices of any triangle.

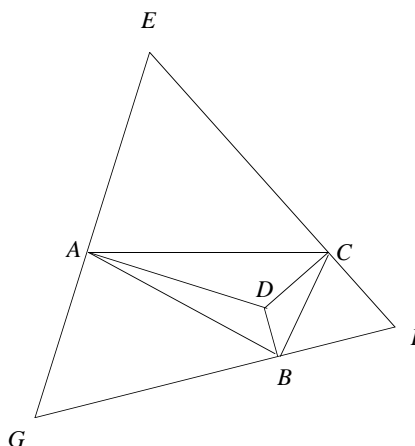
- 1. Write and Reflect** The problem in Investigation 6.7 (“Rich’s Problem”), deals with the sum of the distances to the *sides* of a triangle (an equilateral one at that). The airport problem, however, deals with the sum of the distances to the *vertices* of a triangle. How might these problems be connected? Why might it be helpful to consider the connections?

One way to connect the two problems is to use the same point with different triangles. Start with the airport problem (with Cities A , B , and C). Then, at each city, construct a perpendicular to the road that goes to the airport. Now the sum of the distances to the *vertices* of $\triangle ABC$ is the same as the sum of the distances to the *sides* of $\triangle EFG$.

If $\triangle EFG$ were equilateral, we could now make a connection to Rich's Problem.



2. Show that it is sometimes possible to make $\triangle EFG$ equilateral by moving D around. Is it *always* possible?
3. Show that, if the outside triangle *is* equilateral, then D must be the Fermat point (the alleged best spot for the airport) for the small triangle.

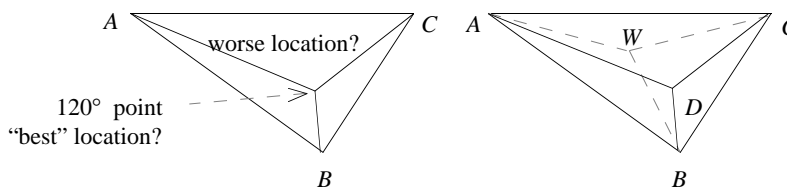


$\triangle EFG$ is equilateral.

So, now we have the situation in which, when D is at the Fermat point for the airport, the big triangle is equilateral. How can we use that to prove the airport conjecture?

Indirect reasoning is another important strategy in mathematics. If we can show that every other point is worse than the 120° point, then we have proved the 120° point is best. Does this sound reasonable?

One way is to use a kind of indirect reasoning. According to the airport conjecture, the sum of the distances from any point *other than* D to the cities is greater than the sum of the distances from D to the cities. Let W be such a point:

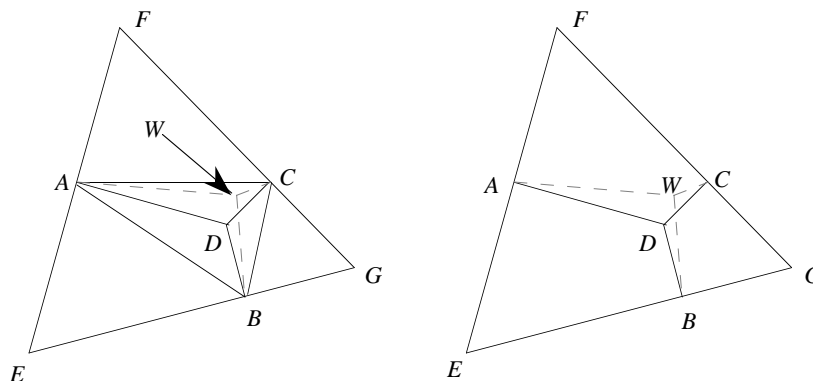


Is W worse than D ?

We want to show that the sum of the distances from D to the three cities is less than the sum of the distances from W to the cities. In symbols, we want to show that

$$WA + WB + WC > DA + DB + DC.$$

4. Reconstructing the big equilateral triangle by sketching perpendiculars to the roads to D at the cities, we have this situation:



The figure on the right ignores the triangle connecting the cities.

Complete the argument to show that $WA + WB + WC > DA + DB + DC$.

TAKE IT FURTHER.....

5. Make a diagram with cities in a triangle that has one angle greater than 120° . Investigate the above construction in this case (this is especially easy if you use geometry software). Is Theorem 6.3 (from Investigation 6.7) any more helpful than Hoffman's proof of the 120° conjecture in deciding where to put the airport?
6. Now you know how to locate the point that minimizes the airport function. Find a simple way to describe the actual *value* of the airport function at the best spot. That is, find a simple way to describe the actual sum of the distances to the cities from the Fermat point.

PERSPECTIVE ON THE STEINER PROBLEM

What is Mathematics? by
Richard Courant and
Herbert Robbins (Oxford
University Press), New York,
1941.

No historical sketch of the problem would be complete without mention of the fact that the Fermat problem has been widely popularized by Courant and Robbins (1941) under the name of "Steiner's Problem". Steiner's contribution to the story is the proof of the theorem contained in this activity. Without ever thinking about airports and without knowing our student Rich, Steiner came up with a proof that connected Theorem 6.3 (from Investigation 6.7) to the airport problem.

Courant and Robbins popularized the more general problem, which allows for the addition of many intersection points. They called these the Steiner points, and they called Fermat's problem the Steiner problem. However, Steiner had considered only one point. Gauss may have been the first to consider more than one Steiner point.

THE RESEARCH CONTINUES TODAY

Frank Hwang is a mathematician working at Bell Laboratories. He was born in Shanghai on August 24, 1940. Two months before his 20th birthday, he earned a B.A. degree in modern languages. In 1962, he passed the TOEFL (Test of English as a Foreign Language) and came to the United States to study. The only problem was that he didn't know what to study. His uncertainty was reflected in one year of studying

International Relations at City College of New York, followed by two years earning an M.B.A. at the Baruch School. While earning his M.B.A., he was required to take a statistics course. This became his next major, and he worked toward a Ph.D. at North Carolina State University.

Dr. Hwang has worked at Bell Labs since 1967. His mathematical interests changed, gradually shifting from statistics to discrete mathematics. He has published about 300 research papers (one tenth in Chinese), written three books (one in Chinese), and edited two books. His major extracurricular activity is bridge, the card game. He has been on the Republic of China's national bridge team several times, the first time when he was seventeen. The pinnacle of his bridge achievements took place in 1969; he was the runner-up at the World Bridge Team Championship.

Dr. Hwang's work at Bell Labs often has put him in contact with the "airport problem." In fact, he coauthored a book (*The Steiner Tree Problem* by Hwang, F. K.; D. S. Richards; and P. Winter; New York: North-Holland, 1992) which gives some of the modern applications of the result and its generalizations. We asked Dr. Hwang to give us a few examples. He sent us the following summary:

Here are some modern developments associated with the Steiner problem:

In 1934, two Czechs, Jarník and Kossler, first studied the Steiner problem in its full generality, (with n points). Unfortunately, their work was only published in a Czech journal and did not receive the attention it deserved; it was not even referenced in the Courant and Robbins book.

A *metric* is a measure of how far apart things are. The distance between two points as measured by the distance formula is one example.

Tonya McLean and Scott Greenleaf used this metric in their work at SMALL. Do you remember the minimal network problems and the farmer's pens problem in Investigation 6.2?

Since 1960, the Steiner problem has picked up full steam, not only as a plane geometry problem, but also for higher dimensions and for other metrics. One metric of special importance is the rectilinear metric, sometimes called the Manhattan metric, which superimposes a rectangular grid through the given points and measures the distance between two points by their vertical and horizontal distances on the grid. This is the metric used in routing the VLSI (Very Large Scale Integrated circuit) chips. The wires interconnecting a given set of electrically common points on the chip are allowed to bend or to meet only in angles of 90° . The two sets of wires, vertical and horizontal, are usually routed in different layers of the chip so that two wires in the same layer maintain a constant distance (wires getting too close can cause electrical problems). An interconnection with minimum length indicates in general an

In this model, length corresponds to cost, not distance.

All these different types of Steiner problems, their theories and methods of solution, are discussed in Dr. Hwang's book, *The Steiner Tree Problem*.

interconnection realized in a small area of the chip, which is desirable in our never-ending effort to shrink the chip.

Sometimes the cost of connecting two points is not proportional to the distance between them. Then we need a graphical model to represent the more general cost structure. For this version of the Steiner problem, the set of given points is connected by various *edges* (line segments), and each edge in the graph can be assigned an arbitrary length which represents the cost of connecting the two endpoints. The absence of an edge between two points either indicates that it is not feasible to connect them physically or that the cost would be exorbitant. Again we want to minimize the total length of such an interconnection.

The Steiner problem also rears its head unexpectedly in biology. For more than 100 years, biologists have attempted to infer *evolutionary trees*. These trees provide information on basic classifications, on when divergence occurred, and on how far apart one species is from another. These questions can be represented by a Steiner problem. The input is a set of species, together with information about each species and relations between the species. The output is a tree (a form of interconnection) that best fits this information. The species are at the leaves of the tree. The internal nodes of the tree, corresponding to Steiner points, are inferred ancestral species.

PROJECTS RELATED TO THE AIRPORT PROBLEM

This investigation gives you some ideas about various projects you can do that are related to the airport problem.

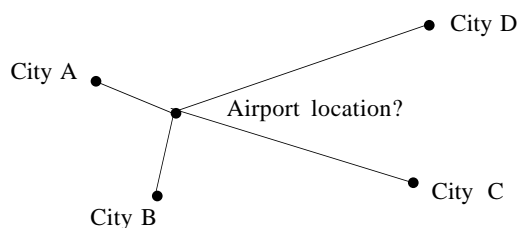
PROJECT 1: MAKE A PRESENTATION

Prepare a presentation for the mayors and city councils of your three cities. Justify your choice of the airport site by showing where the theoretically-best spot is (dazzle them with Hoffman's proof and computer demonstrations), and then explain why you either do or do not recommend this mathematically-determined location. Use and explain the contour plots for the airport function.

PROJECT 2: LOOK AT FOUR CITIES

An interesting conjecture arose from the following question posed in one class that studied the airport problem: *What if there are more than three cities?* The situation is more complicated than with three cities, but you can handle it. Four cities leads to an interesting possibility, and the conjecture leads to another problem altogether.

1. Make a diagram with four cities, and try three airport locations. For each location, measure the distance to each city, and calculate the total distance to the cities. Would contour lines help you?
2. Explain how the models used in the three-city problem could be extended to the four-city case. When using four cities, it seems to make a difference whether you want the vehicles to travel the least total distance, which might keep pollution to a minimum, or you want to require the least road building. It appears that, for this case, minimizing roads and minimizing driving are two very different problems.

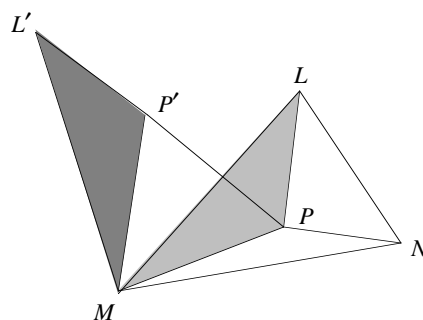


PROJECT 3: INVESTIGATE A RELATED PROBLEM

Minimizing the sum of the distances to a set of points has applications to situations other than locating airports. Investigate how the ideas you've developed and the results you've established in your study of the airport problem can be applied to areas like the methods used by a telephone company to charge people for long distance service and minimal surfaces such as soap bubbles.

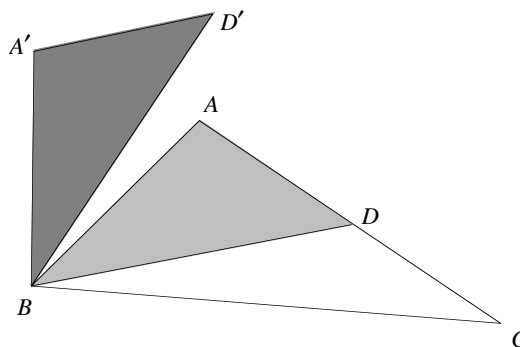
USING IDEAS FROM THE AIRPORT PROBLEM

1. *What makes a proof good or bad? Correct or incorrect? Elegant or awkward?*
These are questions students and mathematicians think about. Write about one or more of the following questions. Be sure to include examples.
 - a. What makes a good proof?
 - b. What are some things that are “illegal” in a proof?
 - c. Several proofs of different types are presented in the module. Choose one that you particularly liked or understood well; write about what makes it a good proof.
 - d. You may have heard someone say, “That is a really elegant proof.” What makes a proof “elegant”?
 - e. Take a statement that was proved in this Student Module, and prove it another way. (Check with your instructor about your choice of statement.)
2. Define each of these terms:
 - rotation
 - center of rotation
 - angle of rotation
3. The figure below shows a 60° rotation of $\triangle LPM$.



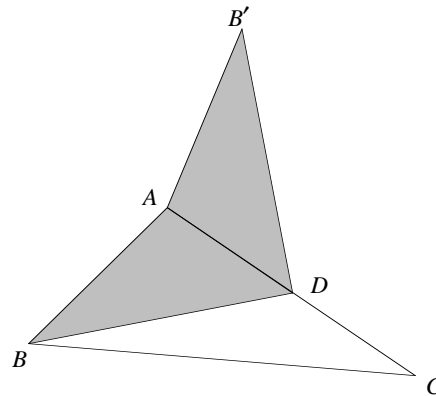
- a. What appears to be the center of rotation?
- b. We were given the angle of rotation (60°). Name all the angles in the sketch that must be 60° . Be sure to explain why each must be 60° .

- c.** Given that rotating a triangle does not change the lengths of its sides or the size of its angles, decide which of the following statements are true. If a statement is false, correct it by changing the measures, points, or segments that make it false.
- i.** $LP = L'P'$
 - ii.** $MP = MP'$
 - iii.** $\triangle MPP'$ is isosceles.
 - iv.** $m\angle MPP' = m\angle MP'P = 60^\circ$
 - v.** $MP + PN = PP' + PN$
 - vi.** $L'P' + PP' + PN = L'N$
- 4.** The following figure also pictures a rotation. Here, $m\angle DBD' = 45^\circ$ and D is the midpoint of \overline{AC} .



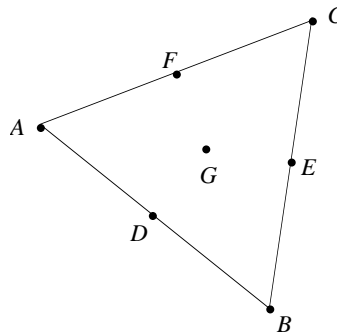
- a.** What has been rotated in this figure?
- b.** What is the center of rotation?
- c.** What is the angle of rotation?
- d.** Name seven or eight pairs of congruent objects in the figure. For example, because D is the midpoint of \overline{AC} , $\overline{AD} \cong \overline{DC}$.
- e.** Suppose that $m\angle ABD = 35^\circ$, $m\angle DBC = 20^\circ$, and $m\angle ADB = 50^\circ$. What other angle measures can be determined? Find as many as you can.
- f.** What if D is the center of rotation and $\triangle ABD$ is rotated 180° ? Draw and describe what you think the picture would look like.

5. Here is another variation on our drawing. Could this be a picture of a rotation? If it is, what are the angle of rotation and center of rotation and what is being rotated? If it is not, explain why.



6. The various “centers” of triangles listed below are also called *points of concurrency*.
- a. Define each of the following “centers,” and provide a simple sketch to show what each one is. (You may need to refer to a mathematics textbook or dictionary.)
- centroid
 - circumcenter
 - incenter
 - orthocenter
 - Fermat point
- b. Which of the points of concurrency listed above would you use in the following situations:
- i. Find the location for an airport so that it would be the same distance from each of three cities.
 - ii. Find a location for an airport so that it would be the same distance from already existing highways that connect each of the cities to the other two.

- c. Write your own question (whether about the airport situation or some other triangular configuration) for which the correct answer is *centroid*.
7. $\triangle ABC$ below is equilateral. Points D , E , and F are midpoints; G is the centroid.



- a. Which other points of concurrency (see Problem 6) does G represent?
- b. Listed below are some objects that might be found in $\triangle ABC$ by connecting any of the given points, A through G . Indicate which of the items can be found by connecting various points. How many of each can you find?
- i. median
 - ii. equilateral triangle
 - iii. isosceles triangle
 - iv. trapezoid
 - v. rhombus
 - vi. kite
 - vii. other convex, 4-sided polygons
 - viii. convex, 5-sided polygons
 - ix. concave, 5-sided polygons
 - x. regular polygon
 - xi. congruent right triangles

Hint: The centroid of a triangle is $\frac{2}{3}$ of the way from each vertex to the midpoint of the opposite side.

8. Suppose that $\triangle ABC$ from Problem 7 is drawn on an x - y coordinate grid, with A at the origin and B at $(4, 0)$.
- What is the slope of \overrightarrow{AC} ?
 - What is the slope of \overrightarrow{BD} ?
 - Write equations for \overrightarrow{BC} and \overrightarrow{AF} .
 - What are the coordinates of points C , D , and G ?
 - What is the length of \overline{AF} ?
9. Relate a strategy used in the Hoffman proof (see Investigation 6.15) to a strategy that might be used to solve the burning tent problem in Investigation 6.2. Explain what is similar and what is different in the use of that strategy for each of the two problems.

THE ISOPERIMETRIC PROBLEM

The answer was to make the floor of the house a square.

It is probably the most famous because it is the most general: it allows you to use *any* shape.

The word *isoperimetric* comes from three Greek word pieces:

iso- means “equal,” as in *isobars*, *isosceles*, *isotope*, *isometric*, and *isomorphic*. (How are isobars and contour lines related?)

peri- means “around.” Look up *pericardium*, and think how *periodical* and *peripheral* might be related.

Meter or *metri-* means “measurement.” You see it in *meter* (as an instrument and as a unit) and *thermometer* and *barometer*. (Remember the *isobars*?) What is a *metronome*? What is *geometry*? Why are *isometric exercises* called that?

In Problem 3 in Investigation 6.3, you looked at the problem of finding the largest rectangular house with perimeter 128 feet. This is an example of what’s called an *isoperimetric problem*. In general, isoperimetric problems involve looking at all geometric figures of a certain type (for example, rectangles) with the same perimeter and then finding one that encloses the most area.

Suppose the house didn’t have to be a rectangle; suppose it could be *any* shape, as long as the *perimeter* of the house remained 128 feet. What shape would you make it?

Or suppose you had a rope of some fixed length, and you joined the ends to make a loop. The loop can be put down on the floor to form many shapes—things that look like polygons, curves, or even figure eights—all with the same perimeter. Which of them has the greatest area? This is the most famous isoperimetric problem. It is known as *the* isoperimetric problem:

PROBLEM *The Isoperimetric Problem*

Of all closed curves with the same perimeter, which one encloses the most area?

You may have already thought a bit about the isoperimetric problem in considering some of the first “For Discussion” questions back at the beginning of this module. The following sections ask for a closer and more careful look at the problem.

1. Make up an isoperimetric problem for triangles.
2. Experimentally—with rope, pencil and paper, a computer, or otherwise—come up with a conjecture for the isoperimetric problem. Explain what you did to convince yourself of your conjecture.
3. How would you go about proving your conjecture? You don’t have to prove it, actually, just make a plan. How would you start? What intermediate results would you try to establish? What techniques would you use?
4. The isoperimetric problem asks which *one* curve encloses the most area. Could there be *more* than one best curve? Could there be *no* best curve at all? Explain.

WHAT YOU WILL DO IN THIS INVESTIGATION

The isoperimetric problem is not an easy one. Experimenting with and conjecturing about which shape maximizes area for a given perimeter is not the hard part. The hard part is to *prove* your conjecture is true.

From your experiments, you may have decided a circle is the best shape. Let's state this as a working conjecture:

CONJECTURE *The Isoperimetric Conjecture*

Of all closed curves with the same perimeter, the circle encloses the most area.

In this investigation, you will study one plan for a proof. You will look at a sequence of partial results which, if they were established, would imply that the conjecture is a theorem. And, you will establish all but one of these partial results. So, by the end of the investigation, you will have a good idea of how the isoperimetric conjecture can be proved.

- 5. Write and Reflect** The role of proof: some people say that they can become convinced of a fact without a mathematical proof. Others say they *never* admit that something is true unless they can supply a proof or logical argument for it. Still others say that, even if they are already convinced of something, finding a proof for it gives them a deeper understanding. Some say that, if they are *sure* of something, a proof that it's *not* true won't change their mind. What do *you* think? Write an essay about the role of proof in the way *you* work. Include some concrete examples.

THE PLAN

The particular plan we're laying out here follows the one described in Courant and Robbins (1941).

Here are the main steps in one plan for establishing the isoperimetric conjecture. Don't worry about completing these steps right now. We'll look at them more closely soon.

Step 1 For a fixed perimeter, show that there is *at least one* curve that encloses the most area.

How do you define
“diameter” in a circle?

Step 2 Given a curve that encloses the most area, show that any line that cuts its perimeter in half also bisects its area. Call such a line of symmetry a “diameter” of the curve.

Step 3 Given a curve that encloses the most area and given one of its diameters, show that, if you pick any point on the curve not on the diameter and draw segments from it to the endpoints of the diameter, the angle you get is a right angle.

FOR DISCUSSION

How do the three steps above imply that a circle encloses the most area for a given perimeter?

The ideas in this investigation, and especially the three steps outlined in “the plan,” took centuries to evolve. (See the historical notes in Investigation 6.20 for an account of this history.) When reading a plan like that above, many people might say, “I’d never think of that. Either mathematics is a collection of facts from out of the blue or it’s done by people much smarter than me.” Neither of these is true.

The truth is that mathematics is an enterprise carried out by a whole community of people, each one contributing small insights and occasional breakthroughs. “The plan” wasn’t created by any one person; it emerged from the work of dozens of mathematicians and from the efforts of teachers and students over the centuries. As an English teacher might say, it’s the result of a great deal of “polish.”

But an important part of learning to do mathematics is learning to *read* and *understand* mathematical expositions like the one in this investigation. Think of this proof as a “finished product,” here for you to study. No one person came up with the design of the gasoline engine, but many people become good mechanics and designers by studying how the finished product works.

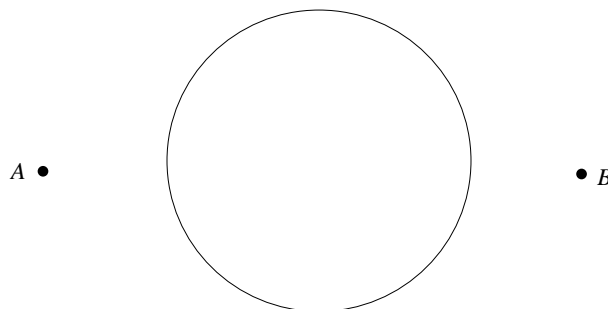
THE STEPS—A CLOSER LOOK

Step 1 As it turns out, this step is the one that can’t be proved using only the methods we have developed so far. This is the step in which you are supposed to show that,

given a fixed perimeter, there *is* a curve of a given perimeter that encloses the most area.

How could there *not* be a solution to an optimization problem? Examples come from the next problem set.

6. Give an example of an optimization problem that has no solution. That is, find a situation where you are asked to minimize (or maximize) something that can't be minimized (or maximized)—there isn't any way to make it as big or as small as possible.
7. Trace the figure below, including *A* and *B*. The object of this game is to draw the shortest possible path from *A* to *B* without touching the circle. Try the game with a partner.



A path can be made up of segments or curves, but you have to be able to draw it without lifting your pencil from the paper. What flexible material could you use to measure the path you drew?

This is one of those problems where you have to read and understand a proof, so read carefully.

If something exists, you can name it anything you like.

8. **Challenge** Many positive numbers are smaller than 1. Examples include $\frac{1}{2}$, $\frac{3}{4}$, $\sqrt{2} - 1$, and 0.897. But is there a *largest* positive number less than 1? Here's an argument which shows that such a largest number can't exist.

Suppose there *were* a largest number smaller than 1; call it x . Since $x < 1$, $1 - x$ is a positive number, so $x + \frac{1-x}{2}$ is a number larger than x .

But x is already the *largest* number less than 1, and $x + \frac{1-x}{2}$ is a number larger than x . So, $x + \frac{1-x}{2}$ must be greater than or equal to 1:

$$x + \frac{1-x}{2} \geq 1.$$

Now use algebra to solve this inequality:

- Multiply both sides by 2. $2x + 1 - x \geq 2$
- Simplify. $x + 1 \geq 2$
- Subtract 1 from both sides. $x \geq 1$

Why do you think this is called an “indirect proof”?

But we started by assuming that x is *less* than 1. How can a number be less than 1 and greater than or equal to 1 at the same time?

- Restate the above proof in your own words.
- Draw a number line showing 0 and 1, and possible positions for x and $x + \frac{1-x}{2}$.
- Why does this argument show that the optimization problem “find the largest number less than one” doesn’t have a solution?

FOR DISCUSSION

Problems 7 and 8 show how optimization problems can have no solutions. What did you and your classmates come up with for Problem 6? Problem 8 shows how you can get into trouble if you assume that there *is* a solution when there isn’t.

But why worry about that in the case of the isoperimetric problem? After all, in some of the earlier problems in this module, you simply located a spot and then showed that every other spot was worse. The key is that you located this best spot without assuming it existed in the first place. In the burning tent problem, for example, you could just reflect your position over the shore and connect what you got to the tent’s position. Something like that *could* happen in the isoperimetric problem, but it *doesn’t*.

- Look at Steps 2 and 3 in the plan for establishing the isoperimetric conjecture. Point out the places where these steps *assume* that there is a best curve.

Because Steps 2 and 3 assume that a best curve exists, the proof of the isoperimetric conjecture isn’t complete until Step 1 is established.

- See if you can make a plan for a proof of the isoperimetric conjecture (different from the one given here) that wouldn’t need to assume that a best curve existed.
-

In 1838, Steiner assumed the existence hypothesis when proving the Isoperimetric Theorem. For more details, see Investigation 6.19.

For the rest of this investigation, we'll assume that Step 1 can be established; we'll call this the "existence hypothesis."

HYPOTHESIS *The Existence Hypothesis*

For any given perimeter, there is a closed curve that encloses the most area.

Assuming the existence hypothesis, we'll derive Steps 2 and 3. The proof of the conjecture will be complete when you prove the existence hypothesis.

.....

WAYS TO THINK ABOUT IT

For a "curve," we'll allow things like polygons and figures made up of both segments and arcs.

Once we agree that there *is* a best possible curve, we can start to figure out some of its properties. Remember, we want to prove the conjecture that the best possible curve must be a circle. So, little by little, we'll rule out curves that aren't circles. What follows will be a little like a detective story: We're on the hunt for a mystery curve, and we'll try to figure out what it is by nailing down its properties.

Maybe you won't prove the existence hypothesis today; maybe you won't prove it this year. It would be nice if you could prove it *someday*.

But it *does* matter that the existence hypothesis *can* be proved. Assuming that something exists and then determining its properties on the basis of these assumptions is a dangerous trap that should be avoided. This is true everywhere, not just in mathematics. It has had some serious consequences in the social sciences. One book that discusses this in depth is *The Mismeasure of Man* by S. J. Gould (New York: Norton, 1981).

-
- 9. Write and Reflect** Explain how the "detective method" described above is a different approach than the one used to solve other optimization problems (like the run-and-swim problem or the burning tent problem).

10. Let x be the sum of all the positive powers of 2:

$$x = 2 + 4 + 8 + 16 + \dots$$

Factor out a 2:

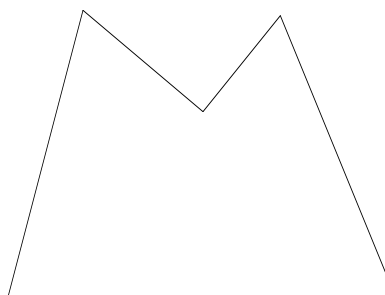
$$x = 2(1 + 2 + 4 + 8 + \dots).$$

The expression in parentheses is just $1 + x$, so

$$x = 2(1 + x).$$

Solve this for x . How can this be?

11. Show that the best possible curve for the isoperimetric problem can't look like this:



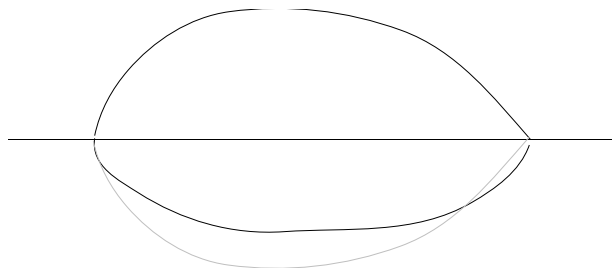
Think of the vertices as hinges.

12. Show that the best possible curve for the isoperimetric problem can't cross itself like a figure eight does.
13. Suppose we knew that the best possible "curve" for the isoperimetric problem is some kind of quadrilateral. What could you conclude about that quadrilateral?
14. Now show that the best possible curve for the isoperimetric problem isn't a quadrilateral.
15. The existence hypothesis is concerned with ensuring that a curve enclosing the maximum area *exists*. Could there be more than one? Give an example of an optimization problem that has several solutions. That is, find a situation for which there is more than one way to maximize or minimize something.
16. **Write and Reflect** Write a clear explanation of what you've shown in Problems 11 through 14. Be sure to explain why what you've shown is true.

If your problem involves geometry, and if your solution is a geometric figure, a congruent figure doesn't count as a "different" solution.

Step 2 We want to show that any line that bisects the perimeter of a best possible curve also bisects its area.

Take a look at any line that cuts the perimeter of a curve in half.



.....
WAYS TO THINK ABOUT IT

This strategy will show up throughout the rest of this module.

Transform a curve in a way that keeps the perimeter the same and increases the area.

.....

If the top part has the same perimeter as the bottom part but has more *area*, you could replace the bottom half by the reflection of the top half. This wouldn't change the perimeter and would *increase* the area.

- 17.** How do you know there *is* a line that cuts the perimeter of the best possible curve in half? Give a method for constructing such a line.

CHECKPOINT.....

18. Define each of the following words.

congruent

symmetry line

isoperimetric

perimeter

diameter

convex

concave

19. State the existence hypothesis in your own words. Why do we need to assume it in order to establish the isoperimetric conjecture? Again, answer in your own words.

20. Make a list of everything you currently know about a curve that encloses the most area for a given perimeter. Give examples of several curves that satisfy all the properties on your list.

If noncircles satisfy your properties, we haven't established the isoperimetric conjecture yet. Why?

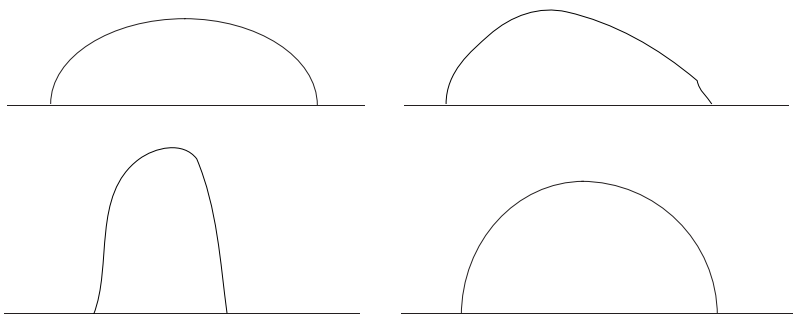
FOR DISCUSSION

Because of Step 2 and Problem 17, we know that there is a line that divides the best possible curve into two pieces with the same area. In light of this, explain why the isoperimetric conjecture is equivalent to the following conjecture.

CONJECTURE *The Reduced Isoperimetric Conjecture*

Take a line ℓ . Of all the curves that have two endpoints on ℓ and have the same perimeter, the one that encloses the most area is a semicircle whose diameter lies along ℓ .

The reduced isoperimetric problem is very reminiscent of the “fence against the wall” problem (Problem 6 of Investigation 6.3): You have a long wall, and a fixed length of fence. You want to make a pen from the fence using the wall as one side of the pen. What shape fence maximizes the area of the pen? The next pictures show some possible shapes for the pen (all the curved parts are supposed to have the same length):

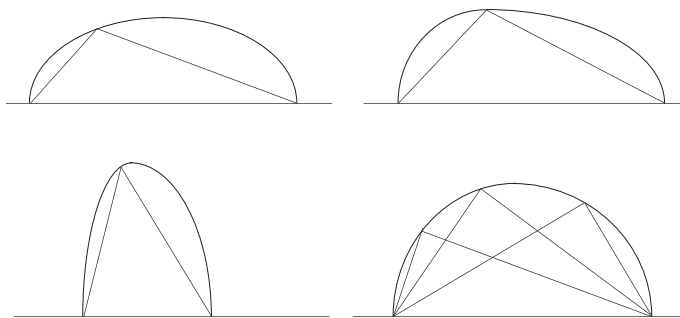


We want to show that the semicircle answers the question.

- 21.** Can you use the same method you used to solve the fence problem (Problem 6 in Investigation 6.3) to solve the reduced isoperimetric problem? Why or why not?

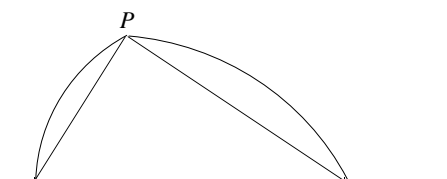
Step 3 Here the goal is to show that if you pick any point on the best possible curve and draw segments from it to the endpoints of a diameter, the angle you get is a right angle.

- 22.** In light of the reduced isoperimetric conjecture, we only have to consider half of a curve. If the condition in Step 3 is met, does this guarantee that the curve will be a semicircle? Explain.

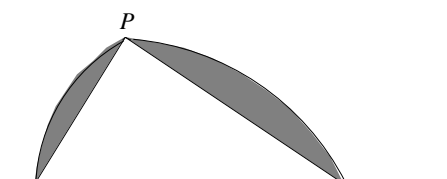


.....
WAYS TO THINK ABOUT IT

One way to think about establishing Step 3 is to imagine that you have a candidate for the best curve. Draw a diameter and connect the endpoints of the diameter to some point P on the curve.



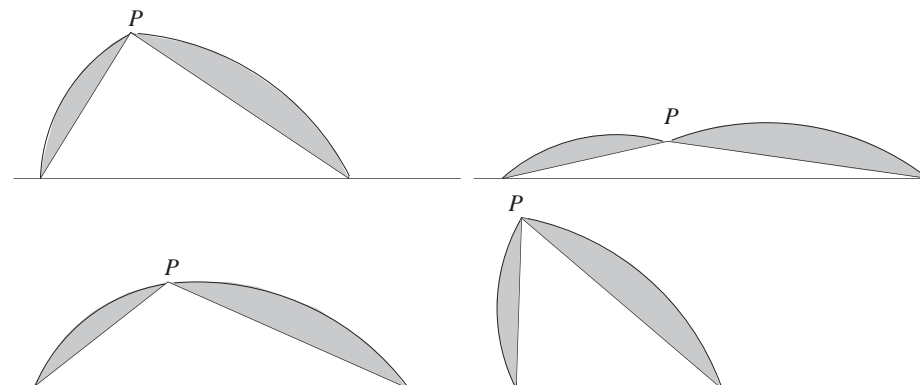
The idea is to show that the angle at P should be a right angle. If it isn't, you can show how to build another curve with the same perimeter and *more* area. Imagine that the pieces between the segments and the curve are made of cardboard.



They look somewhat like crescent moons.

Also imagine that there's a hinge at P , so that the size of the angle can change. As the angle opens and closes, the pieces of cardboard don't change at all:

You can turn this thought experiment into a real experiment using geometry software.



Since the cardboard pieces don't change, the perimeter will remain the same. Yet the area changes. The question that arises is the one you are asked in Problem 23 below.

.....

You might want to look at Problem 1 in Investigation 6.3.

- 23.** For which angle is the area a maximum? Explain.
- 24. Write and Reflect** You now have everything you need to prove the isoperimetric conjecture (assuming the existence hypothesis). Write out a careful proof in your own words, using examples and pictures, and explaining every step.

We can now state the conjecture as a theorem:

THEOREM 6.5 *The Isoperimetric Theorem*

Assuming the existence hypothesis, the circle is the curve that encloses the most area for a given perimeter.

A PROBLEM WITH A LONG HISTORY

The source of this legend is Virgil's *Aeneid*.

According to legend, Phoenician Princess Dido was fed up with the scandals and divisions within her royal family. One day, having become so unpopular that she feared for her life, she gathered together some of her still faithful friends and relatives and fled her home in Tyre. She boarded a ship to North Africa and eventually landed just west of Libya on the shores of modern-day Tunisia, where she decided to set up a new home. The princess signed a contract with local landowners to buy a small plot of land there. Thinking that they could get the best of this newcomer, the landowners wrote into the contract that she'd get only as much land as she could surround using the hide of a bull. But the princess was much shrewder than they.

Princess Dido cut the bull's hide into very thin strips, practically like threads. Using the Mediterranean Coast as a straight boundary of her land, she laid out the thin strips of bull's hide in the shape of a semicircle. Within this plot of land she built a new city to be called Carthage.

This ancient legend is not a verified history. But Carthage was quite real, and the story of its founding over 3000 years ago—true or not—is the earliest known instance of stating and attempting to solve the isoperimetric problem.

1. State in a precise way the optimization problem that Dido is said to have solved.

The next recorded development in the history of the isoperimetric problem took place around 330 B.C. in the writings of the great philosopher Aristotle. All we have left is a tantalizing glimpse of the problem in a single sentence found in Aristotle's *Posterior Analytics*: "It is the physician's business to know that circular wounds heal more slowly; it is the geometer's business to know the reason why."

2. Connect Aristotle's statement with the isoperimetric problem. Be the geometer and explain why shape might have anything to do with the rate of healing. Why, if indeed it is true, might the circular wound heal most slowly?

Almost a century later, around 240 B.C., Archimedes solved a variety of isoperimetric problems. Archimedes lived in Syracuse, then Greek but now a part of Italy. He was well known, both in his own time and today, for his accomplishments in science and engineering as well as in mathematics.

Why would a sphere maximize volume for a given surface area? (Think about balloons.)

We know with certainty that Archimedes proved a three-dimensional analog of Dido's version of the isoperimetric problem, demonstrating that the hemisphere contains the greatest volume of all spherical segments with a fixed surface area. Some say—

although it is not known for sure—that Archimedes also proved the theorems that were later referred to by Ptolemy.

Writing around A.D. 150, Ptolemy stated in his book *Syntaxis* that “of the different figures that have equal perimeter, the more polygonal ones are greater, so that the circle is greater among the plane figures, while the sphere is greater among the solid ones.” He did not include a proof of this proposition—at least in his works that survive today—so we cannot be sure whether he had one or not.

3. In modern terminology, figures are either polygonal or they are not. One cannot be “more polygonal” than another. Judging from the conclusion that Ptolemy came to, what must he have meant by “more polygonal”?

See *The Ancient Tradition of Geometric Problems* by Wilbur Richard Knorr (Boston: Birkhauser, 1985).

Reportedly, another mathematician, Zenodorus, wrote an entire book called *On Isoperimetric Figures*, in which several important theorems were proved. One was that of all polygons with a fixed perimeter and fixed number of sides, the *regular* polygon has the greatest area. A second was that a circle with that perimeter (circumference) has an even greater area. A third was that of segments of circles with a fixed perimeter, the semicircle has the greatest area—a variant of Dido’s problem.

4. Three of Zenodorus’s theorems are listed above. For each one, illustrate what it says with a specific example. Use pictures and words for your examples.
5. Explain how the theorem about the fixed-perimeter segments of circles is a variant of Dido’s problem.
6. Do these theorems imply that Zenodorus proved the isoperimetric conjecture?

What is a *segment* of a circle?

Historical records show no further development of the isoperimetric problem for many centuries. It appeared again in 1697—which suddenly seems very close to the present when we look at the scope of the history of Western mathematics! In May of that year, the Swiss mathematician Jakob Bernoulli constructed an especially complicated isoperimetric problem.

After the move from ancient Greece to modern Switzerland, the saga of the isoperimetric problem remained quite active. In the 1700s, a number of mathematicians were making use of the recent development of calculus to analyze a great many optimization problems. But the Swiss mathematician Jacob Steiner was working on solving such problems *without* calculus. He attempted to provide rigorous proofs of the isoperimetric theorem. Steiner’s proofs were indeed rigorous and accurate—but with one small difficulty. Steiner assumed that a curve containing maximum area *exists*. (This is our “existence hypothesis.”)

Despite other mathematician's attempts to persuade Steiner that the proofs were thereby incomplete, he continued to claim that the existence of a curve containing maximum area was "self-evident." You, yourself, have seen what trouble *that* can cause (recall, for example, Problems 7 and 8 in Investigation 6.2). Finally, in 1842, Steiner admitted that this unstated assumption should at least be explicitly stated (as we did in this investigation), and he added a comment that "the proof is readily made if one assumes that there is a largest figure."

This can be generalized to other polygons—the one with the greatest area is regular.

In his 1842 papers, Steiner provided a rigorous proof of another variant of the earlier claims of Zenodorus: that the equilateral (that is, regular) triangle has the greatest area of all triangles with the same perimeter.

Over the years from 1870 to 1909, the hole in Steiner's 1838 proof was gradually filled by the work of a variety of mathematicians who proved that a curve that contains maximum area must indeed exist. The first complete proof dates to 1901. But the story isn't over. The isoperimetric problem led to other questions, and the research continues today.

- 7. Write and Reflect** Write an essay about what you learned about the isoperimetric problem in this investigation.

RELATED QUESTIONS

There are many questions suggested by the isoperimetric theorem. For example:

Why is this a related question?

See Problem 4 of Investigation 6.3.

- 8.** The theorem says that, for a given perimeter, the circle "fences in" the most area. What about the related question: Of all the curves that enclose a fixed area, which one has the smallest perimeter?
- 9.** Suppose you had to fence in an area of 100,000 square feet, and the cheapest way to install the fence was to use straight segments that go either from north to south or from east to west. What is the size and shape of the fence enclosing the area that has least perimeter?

How is this different from Problem 28 in Investigation 6.2?

Here is another problem suggested by the isoperimetric theorem:

Suppose a farmer has two kinds of animals and keeps them in two connected fenced-in pens. The fence must be replaced, and the farmer wants to rebuild the pens subject to the following constraint: the shapes of the pens can change, but the area of each pen must stay the same. What shapes for the two pens will minimize the amount of fencing?

The question can be stated more precisely as the “two-area problem.”

PROBLEM *The Two-Area Problem*

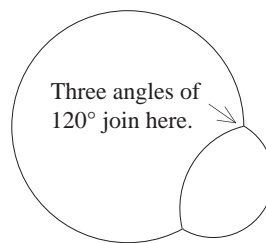
How can you enclose two given areas with the least perimeter?

FOR DISCUSSION

What are some possible solutions to the two-area problem? How would you begin to work on it? What are some special cases that might get you started?

The results of these students were published in the *Pacific Journal of Mathematics*, vol. 159, No. 1, in 1993.

Surprisingly, the two-area problem wasn’t settled until 1990, when a group of college students worked on the problem as part of the summer program called SMALL. The students showed that the best way to enclose the two areas is to use the two-dimensional version of the “standard double bubble,” a figure that is made up of three arcs of circles. (You could also describe it as two circles that have collapsed, or combined, into one compound figure.) In such a figure, the three arcs all meet at 120° angles:



1. You know how to measure angles between lines. What might “the angle between two arcs of a circle” mean?
2. Construct a standard double bubble using geometry software or figure out how to make a rough sketch of one by hand.

The two-area problem suggests the three-area problem:

PROBLEM *The Three-Area Problem*

How can one enclose three given areas with the least perimeter?

What do you think a standard triple bubble looks like?

In 1992, another team of students and their advisor at SMALL found the solution of this problem (assuming connected regions) to be the two-dimensional version of the “standard triple bubble.”

What do you think the “one volume problem” is? What would be a reasonable solution?

Another question suggested by the two-area problem is the two-volume problem:

PROBLEM *The Two-Volume Problem*

How can you enclose two given volumes in three-dimensional space with the least surface area?

These questions about the optimal enclosures for various connected areas and volumes represent another instance of a set of problems with a long history for which communal effort was involved in their solution.

In 1995, two mathematicians announced a proof that the double bubble is the most efficient for the special case of enclosing two *equal* volumes. An undergraduate who had been involved with the 1992 team at SMALL and who had later narrowed down the possible solutions for the case of equal volumes, continued working on other volume ratios as a graduate student. The findings introduced new mathematical techniques which can be put to use in the field of geometry and optimization theory. Perhaps a proof of the general solution for the minimal enclosure of *any* two given volumes will be discovered.

These findings also shed light on interesting practical applications, most visibly wherever efficient containment is necessary. One suggestion is in engineering, particularly aerospace engineering: the double bubble might be used to minimize the mass of satellite or flight tanks that need to hold two gases or liquids that must be stored

separately—ideal for something like liquid oxygen and liquid hydrogen. Such a container could minimize space used as well as mass.

PERSPECTIVE ON STUDENT RESEARCH EXPERIENCE

In this essay, a student describes original mathematics research in which he participated at a summer program for college students. His work was closely related to problems you have solved in this module.

Scott applied and was accepted to an undergraduate research program in mathematics and physics for the summer before his senior year at a college in Maine. While there, one of the projects he worked on included a related isoperimetric problem.

See Investigation 6.2 to read Tonya McLean's essay about the same program and to take a look at a few problems similar to those the students worked on.

The following is Scott Greenleaf's description of his experience in 1993 at an undergraduate research program. Scott was part of the team that solved the two-area problem specifically for the "taxicab" perimeter—meaning the perimeter must consist only of horizontal and vertical line segments. (See the farmer and the two connected pens problems in Investigation 6.2.)

"During my junior year as a mathematics major, I decided that I wanted to write a senior year thesis. At my school, students in many majors are required to complete a thesis. For mathematics majors, a thesis is optional, but I viewed extended independent work in my field as a challenging experience very different from any offered in a typical course.

"Many topics interested me, but I didn't know what mathematical *research* was about. I just assumed that a thesis in mathematics would involve writing about an interesting branch of mathematics discovered long ago. Like many people, I thought of mathematics as something that was basically "finished"; I didn't realize that there are people actively involved in discovering new things in mathematics today.

"To get ideas for a thesis and to have a more meaningful summer than I would have bagging groceries, I decided to look for research opportunities for undergraduate mathematicians. I was pleased to find that there were several programs to choose among, all over the country, and that descriptions of many of these programs were posted in the mathematics department lounge at my school. I applied to one and was accepted.

"The program gave me my first experience in looking at open questions in mathematics. Before the summer I spent in the program, I thought that any unsolved questions in mathematics are unsolved because they are too difficult for anyone to solve. However, I soon learned that there are mathematical ideas that haven't been examined very closely and there are many interesting discoveries one can make with some mathematical background (sometimes no more than high school mathematics) and a little cleverness or perseverance.

“Each of the students in the program was a member of one or two of the eight groups that comprised the program. One of the groups that I joined was the Geometry Group. Our group examined two main questions. One problem was to find the shape enclosing two regions of prescribed area while minimizing the perimeter.

3. What shape encloses such an area and minimizes the perimeter if there are no restrictions on the type of shape?
4. What shape encloses such an area and minimizes the perimeter if the shape is created using only horizontal and vertical paths?
5. Why do you think the perimeter in Problem 4 is sometimes called the “taxicab” perimeter?

“The other problem was to find the smallest surface joining a pair of semicircular wire frames connected at their centers. To get ideas, we would dip wire frames into soap solution and gaze at the resulting soap film shapes.

“I was also a member of the Minimal Surfaces Group. The Minimal Surfaces Group thought about this question: Given a network that is *approximately* the shortest one connecting a set of points, can we find out how close this network is to the one which is *truly* the shortest? (Investigation 6.2 presents some introductory problems we looked at in analyzing the general formulas for that problem. The shortest network connecting three points is the subject of the third section of this module, “The Airport Problem.”) To find the shortest network connecting more than three points, a computer is needed! Even then, it might take an unreasonably long time to find an exact solution. So computers often are asked only to provide approximate solutions. We were searching for a general formula to tell us how good those approximate solutions are.

“Before I participated in this summer program, I didn’t really enjoy group projects. I would become impatient and had a tendency to want to do all the work myself, because in most of *my* mathematics classes, we were given problems which had a predictable method of solution. In the program, however, we were dealing with problems that no one had really explored, so we had to consider several possible ways to find a solution. Thus, it was much more helpful to be able to discuss ideas with others than to isolate ourselves. We’d meet on our own and with our faculty advisor, talk about what we had discovered and about related research we’d found. Often, when our groups met, someone would bring up something that she was thinking about, and someone else would mention something related he had pondered, and by the end of the meeting, we had all kinds of new ideas that each of us was eager to sit down and think about on our own.

One of the best ways to understand something is to explain it to someone else.

This student's work was presented at a meeting of the Mathematical Association of America in Newport, Rhode Island in 1994. The results from the student work by The Geometry Group was published in the *Journal of Geometric Analysis*.

"One important part of our experience was presenting our research to people outside our group. At the beginning of the summer, the groups explained to everyone the mathematical ideas they would be looking at for the rest of the summer. Then, at the end of the summer, each group presented its results to everyone involved in the program. In addition, each undergraduate was asked to give a talk about his or her research at a conference sometime in the next year. Explaining mathematical ideas to my peers really aided my understanding. Because I had to help familiarize others with these concepts, I had to be very comfortable with the ideas myself.

"The last two weeks of the program were the busiest. As the program wound to a close, the groups focused on writing up their progress for the summer rather than thinking about new research. This was a very hectic time, but it was very satisfying to see the finished products come together.

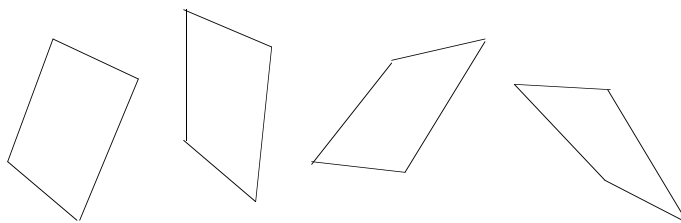
"I am very glad I had the opportunity to take part in this program. The experience gave me a whole new perspective on mathematics. I now realize that mathematics is a continuously growing field, rich with new ideas. It is very empowering to feel that in some way I am part of the frontier of mathematics. I very enthusiastically encourage anyone who might be interested in a summer research program to give it a try. I am sure you will be glad you did."

TAKE IT FURTHER.....

Such a circle is called the *circumcircle* of the triangle. Its center is the triangle's *circumcenter*.

6.
 - a. Show how to construct a circle that passes through the vertices of any given triangle.
 - b. Can you pass a circle through the vertices of any quadrilateral? Explain.
7. A polygon is called *cyclic* if its vertices lie on a circle.
 - a. Show that every triangle is cyclic.
 - b. Draw a picture of a quadrilateral that isn't cyclic.
 - c. Find a way to tell whether or not a quadrilateral is cyclic just by looking at its angles.

8. Suppose you have four sticks of different lengths joined together to make a quadrilateral with hinges at its corners. You can make all kinds of quadrilaterals with most sticks:



Maybe these quadrilaterals
all have the same area.
Could that happen?

Which one encloses the most area?

***MODULE OVERVIEW AND
PLANNING GUIDES***

ABOUT THE MODULE	T₂
MAIN TIMELINE PLANNING CHART	T₃
ALTERNATE TIMELINES	T₆
ASSESSMENT PLANNING	T₇

ABOUT THE MODULE

This module focuses on optimization problems—finding maxima and minima—but no calculus is used. Some algebraic manipulations are necessary, but most of the problems can be solved with strictly geometric techniques. The module is problem-based. You will notice that many of the notes in this guide refer to specific problems and the outcomes from students’ investigations of them.

Optimization problems are traditionally reserved for calculus. And yet, while most of the problems in this module can indeed be solved with calculus, you may be surprised at how approachable they are with geometric methods, and at how elegant the geometric solutions can be. Moreover, these problems are specifically designed to help students get at the big ideas behind optimization problems—thinking about extreme cases and boundary conditions, and using a mixture of deduction, experimentation, and reasoning by continuity. Thus, they provide the context for the kind of thinking teachers of mathematics (and specifically of calculus) want from their students. In the “Mathematics Connections” section of the Teaching Notes, however, you will find that some of the notes about mathematical connections include connections to calculus.

This module is ideal for:

- an advanced geometry class
- the final 3–6 weeks of a year-long geometry class
- a precalculus class
- a “special topics” class for juniors or seniors in high school
- a course for preservice teachers who have already studied calculus

The module is broken into four sections. The first is an introduction to optimization and some useful geometric techniques. The last three are each extended investigations of a particular optimization problem.

MAIN TIMELINE PLANNING CHART

To do the complete module, you will probably spend 9–12 weeks of class time (depending on how much of the geometry your students already know, and how much is developed in the course of investigating the problems).

Investigation	Description	Key Content	Pacing
6.1 Introduction to Optimization	This is a reading investigation that contains a couple of discussion problems to introduce the meaning of the word <i>optimization</i> . It gives students an idea of what represents an optimal solution and the kinds of problems that call for one.	<ul style="list-style-type: none"> • meaning of <i>optimization</i> • numerical optimization problems 	1–2 days
6.2 Making the Least of a Situation	This investigation focuses on minimizing distance: from a point to another point, from a point to a line, and along other constrained paths. It includes an optional reading activity (with problems) about undergraduate students doing research in mathematics.	<ul style="list-style-type: none"> • straight lines • perpendicular bisector • reflecting a point over a line • minimizing distance 	4–5 days
6.3 Making the Most of a Situation	This is an introduction to the isoperimetric problem: for all curves of a given perimeter, which one maximizes area? It focuses on the question for quadrilaterals and other polygons. It also explores using circles to maximize angles.	<ul style="list-style-type: none"> • maximizing area • square as maximal rectangle • inscribed angles • ellipses 	4 days
6.4 Contour Lines and Contour Plots	This investigation explores functions on the plane ($\mathbb{R}^2 \rightarrow \mathbb{R}$) and the use of contour lines to find minimal values for these functions along a path. It includes some formality about functions that could be skipped by classes of less-experienced students (geometry rather than precalculus, for example) or by classes taking a shorter path through the module.	<ul style="list-style-type: none"> • function • ellipses, circles, arcs • contour lines and surface plots 	4–7 days

Investigation	Description	Key Content	Pacing
6.5 Using Optimization	All of these problems might be used together as an assessment after students have completed Investigations 6.1–6.4. Alternatively, each of Investigations 6.2–6.4 includes notes on when some of these problems might fit in as homework or assessment.	<ul style="list-style-type: none"> homework and assessment problems 	resource
6.6 Introduction to the Magical Mix	This is a reading activity designed to set the tone for the extended investigation.		partial day
6.7 A Student Outsmarts the Test	This investigation introduces Rich’s problem (what is the sum of the distances from a point inside an equilateral triangle to the triangle’s sides?), and develops two proofs of the result.	<ul style="list-style-type: none"> reasoning by continuity functions proof 	5 days
6.8 Variations on a Problem	First, students are asked to come up with variations on Rich’s problem by changing one or more of the problem’s elements. Next, students explore one possible variation on Rich’s problem: what if the point can be anywhere in the plane, not just inside the equilateral triangle? Finally, students explore the variation in which the triangle is not equilateral.	<ul style="list-style-type: none"> extending problems asking “what if” and “what if not” changing the domain of a function contour lines maximum and minimum values of a function analyzing proofs 	6–8 days
6.9 Making Connections in Mathematics	This is a reading activity that connects the investigation of the past few investigations to research done at Bell Laboratories.	<ul style="list-style-type: none"> connections to one mathematician’s work 	1 day
6.10 Using the Magical Mix	The problems are suitable for homework and assessment. They might all be used together as an assessment after students have completed Investigations 6.6–6.8. Alternatively, each of Investigations 6.7 through 6.8 includes notes on when some of these problems might fit in as homework or assessment.	<ul style="list-style-type: none"> homework and assessment problems 	resource

Investigation	Description	Key Content	Pacing
6.11 Getting Started	This investigation presents students with the airport problem (how to find the best location for an airport to be shared by three cities). Students select three cities on which to base their investigation.		1 day
6.12 What is Meant by <i>Best</i>?	This investigation asks students to consider various mathematical meanings for “best spot,” including “fair” (equidistant from each of the three cities) and “environmental” (minimum total distance from the three cities).	<ul style="list-style-type: none"> • continuous variation 	1–2 days
6.13 Special Cases and Models	Students are shown some good ways to explore problems, including simplifying the problem and exploring special cases, making a mechanical model, and constructing a computer model. The goal is to use reasoning by continuity in developing a conjecture about the airport problem. You may want to use one, two, or all three of these approaches.	<ul style="list-style-type: none"> • simplifying problems • mechanical models • computer models • continuous variation • contour lines 	1–3 days
6.14 Testing the Conjecture	The 120° conjecture is stated explicitly. Students develop methods to locate the 120° spot.	<ul style="list-style-type: none"> • conjecturing • 120° spot or “Fermat point” 	1–2 days
6.15 Establishing the Conjecture	This investigation presents two proofs of the airport conjecture: one that is carefully worked out, and one for students to work through.	<ul style="list-style-type: none"> • proof • rotation • straight line as minimal distance 	6 days
6.16 The Airport Revisited	This investigation connects the airport problem to Rich’s problem (from Investigation 6.7), providing a third proof of the airport conjecture.	<ul style="list-style-type: none"> • connections to previous work • proof using previously-established results 	1 day
6.17 Projects Related to the Airport Problem	Three student projects are suggested.		resource

Investigation	Description	Key Content	Pacing
6.18 Using Ideas from the Airport Problem	These problems are suitable for homework and assessment. They might all be used together as an assessment after students have completed Investigations 6.11–6.16. Alternatively, each of Investigations 6.12–6.16 include notes on when some of these problems might fit in as homework or assessment.	<ul style="list-style-type: none"> homework and assessment problems 	resource
6.19 The Isoperimetric Problem	With guidance, students create a proof that for a fixed perimeter the circle is the curve that maximizes area.	<ul style="list-style-type: none"> proof area and perimeter convex and concave symmetry 	6 days
6.20 A Problem with a Long History	This investigation contains essays on the history of the isoperimetric problem as well as related problems for students to solve.	<ul style="list-style-type: none"> historical connections 	3 days
6.21 The Research Continues Today	This investigation presents two-area, three-area, and two-volume problems (all related to the isoperimetric problem). It includes an essay about an undergraduate student's experience doing mathematics research.	<ul style="list-style-type: none"> extending problems area, perimeter, surface area, and volume connections to modern mathematics research 	3 days

ALTERNATE TIMELINES

We offer here four alternate paths through the module, each with a specific emphasis, and all considerably shorter than the main timeline plan. If the full 9–12 weeks is too long for your class, we hope one of these alternatives will suit your needs. These four were chosen to meet the most frequent requests from field test teachers.

Optimization Introduction

• 6.1–6.5 (3–4 weeks)

Introduction to Geometric Optimization (3–4 weeks)

This is our shortest timeline; it's a good choice for end-of-year geometry (or integrated mathematics) classes. This plan has less of a focus on functions and proofs, and more of a focus on problem solving and reasoning strategies.

One Investigation

- 6.1–6.5 (5–6 weeks)
- Choose one of the last three sections: 6.6–6.10, 6.11–6.18, or 6.19–6.21.

Two Extended Investigations

- 6.1–6.5 (3 weeks)
- Choose two of these three sections: 6.6–6.10, 6.11–6.18, or 6.19–6.21 (2–3 weeks each)

Proof-Focused

- 6.1–6.5
- 6.6
- 6.7
- 6.8
- 6.12 (1 day)
- 6.13 (1 day)
- 6.14 (1 day)
- 6.15
- 6.16
- 6.19

One Extended Investigation (5–6 weeks)

Spend three weeks on Investigations 6.1–6.5 and the remaining 2–3 weeks on *one* of the last three sections of the module. You may want to have your students read the introduction to each section, where the problem is stated, to decide which of the three problems appeals to them the most. This is a good plan to follow if you want your students to get a taste for an extended mathematical investigation but do not have 9–12 weeks to cover the whole module.

Two Extended Investigations (7–9 weeks)

The class could cover two of the three extended investigations (Investigations 6.6–6.10, 6.11–6.18, and 6.19–6.21). Spend 3 weeks on Investigations 6.1–6.5, and then 2–3 weeks each on two of the extended investigations. The second and third sections are closely related and are connected explicitly in Investigation 6.16, so they would make a good choice for this plan.

Focus on Proof (8 weeks)

This timeline is for those who wish to take the fullest possible advantage of the module’s compelling offerings on proof. Students will work through all of the problems in Investigations 6.1–6.5 and briefly investigate each of the three extended problems. Less time spent on the investigations will leave more time for a heavier focus on the (sometimes multiple) proofs of each result and for comparison and reflection on the various methods of proof.

ASSESSMENT PLANNING

Suggestions are provided below for different types of assessments. You will not want to use all of these, but rather pick and choose ones with which you are comfortable or with which you want to experiment.

What to Assess

- The student knows how to find minimal distances: from point to point (using a straight line), from point to line (using a perpendicular), and from point to a line or other boundary and then to a second point (using the “reflection principle”).

- The student knows that for a given perimeter the square maximizes area over a rectangle. The student can use cutting arguments or area calculations to compare areas of other figures.
- The student understands that contour lines are curves of fixed value for a function on the plane and can explain how to use them and tangency to solve optimization problems.
- The student can identify circles as curves of constant distance from a single point, ellipses as curves of constant distance from two points, and arcs of circles as curves of constant angle between two points.
- The student can explain the idea of continuous variation in terms of “small changes in input produce small changes in output.” The student reasons about extreme cases (or boundary cases) in solving problems.
- The student pursues an extended investigation: defining a problem clearly, investigating it, conjecturing, proving a result, and exploring extensions.
- The student understands the difference between an initial conjecture, a conjecture supported by *a lot* of evidence, and a theorem that has been proven.
- The student can use reasoning about a continuously varying system to find an approximate solution to a problem. Students investigate special and extreme cases of a problem.
- The student can locate the Fermat point of a triangle and explain why for most triangles it minimizes the sum of the distances to the three vertices.
- The student can articulate the result of the isoperimetric theorem.
- The student can follow a logical argument and, with guidance, construct his or her own proofs. The student can, in specific cases, identify what is necessary and sufficient for a complete proof.

Notebooks

Throughout the entire module, students should be required to keep a notebook that contains:

- daily homework and other written assignments
- a list of vocabulary, definitions and theorems that emerge during classwork and homework

QUIZZES AND JOURNAL ENTRIES

Investigation	Journal Suggestion or Presentation	Quiz Suggestion
6.2	<i>Checkpoint</i> Problem 14: Describe strategies for minimizing distance.	Use any of Problems 3, 4 and <i>Checkpoint</i> Problems 11–14.
6.3	Problem 4: Analyze a cutting problem. <i>Write and Reflect</i> Problem 16: List and illustrate strategies for solving optimization problems.	Use any combination of <i>Checkpoint</i> Problems 10, 11, 13, and <i>Take It Further</i> Problems 23 and 24.
6.4	<i>Write and Reflect</i> Problems 25 and 26: Explain how to draw a contour plot for a function and how contour plots can be used to solve optimization problems.	Use <i>Checkpoint</i> Problem 40 and <i>Take It Further</i> Problem 41. Solve by using contour lines.
6.5	As a final assessment for Investigations 6.1–6.5, have students choose one of the following sets of problems to investigate and to present their findings to the class. If students work in groups, ask for individual write-ups in addition to the group presentation. This is a good opportunity for students to start finding ways to conduct experiments and to present their findings. 6.2 Problem 28: There are four different two-area/minimal perimeter problems. Students should find and articulate general methods for reducing and minimizing total perimeter for two enclosed adjacent spaces, not just solve the set of problems given. 6.3 Problems 18–21: To extend the investigation about maximizing area for rectangles and start students thinking about the more general Isoperimetric Theorem, have students make general conjectures about maximizing area. For example, students might find that increasing the number of sides of a regular polygon while holding perimeter fixed will increase the area.	

Investigation	Journal Suggestion or Presentation	Quiz Suggestion
(6.5)	Problems 12–14: These vary in difficulty and are best explored with geometry software. Outcomes should include descriptions of the method of investigation and any general conjectures.	
6.7	<p><i>Write and Reflect</i> Problem 2: What does “reasoning by continuity” mean?</p> <p>Problems 6 and 7: Students should work through two proofs of the solution to Rich’s problem and present one of the proofs to the class, explaining details and answering questions.</p>	
6.8	<p>Problems 1 and 2: Students select and investigate a variation on Rich’s problem of their own devising. They present the statement of the problem, method of investigation, conjectures, and work toward proof of their conjectures.</p> <p>Problems 9–11: Students investigate the contour lines of the function that measures the sum of the distances from any point in the plane to the three sides of an equilateral triangle, and attempt to prove that the value of the function is constant along the curves. This is a difficult proof and may be best done by a group with a good background in writing proofs.</p> <p><i>Write and Reflect</i> Problem 15: Students investigate the proofs of the solution to Rich’s problem to find what goes wrong when the triangle is not equilateral. The answer provides ideas about where the value is a maximum and where it is a minimum.</p> <p>Problems 18 and 19: Students investigate problems similar to the extension of Rich’s problem to a nonequilateral case. They make conjectures about other functions on a triangle and try to prove them. Students might want access to geometry software.</p>	

Investigation	Journal Suggestion or Presentation	Quiz Suggestion
6.10		Test 6.6–6.10: Include Problems 9 and 12 from Investigation 6.7. Explain why reasoning by continuity helps to solve Rich's problem. Also include a few questions comparing Rich's problem with the extensions. For example, show four pictures of triangles and points D : one equilateral triangle with the point inside, one equilateral triangle with the point outside, one scalene triangle with the point inside, and one scalene triangle with the point outside. Questions: <i>True or False—In each case, the sum of the distances from D to the sides of the triangle is equal to the height of the triangle. If false, explain why. Can you approximate the sum, saying whether it is greater than or less than the height of the triangle?</i>
6.12	<i>Write and Reflect</i> Problem 7: Where should the airport go? (Initial conjecture)	
6.13	<i>Write and Reflect</i> Problem 18: Where should the airport go? (Second conjecture)	State your conjecture as clearly as possible. What are the special triangles where your conjecture doesn't work?
6.14	Problems 9 and 10: Revisit ideas on contour lines in a new context.	
6.15	<i>Write and Reflect</i> Problems 7 and 15: Students speculate about where Hoffman and Toricelli got their ideas. Students examine a proof or a construction in order to think about where the idea may have come from.	Problems 3 and 4

Investigation	Journal Suggestion or Presentation	Quiz Suggestion
6.17	<p>This investigation provides three ideas for final presentations. Students choose one to work on and then present their results to the class. In addition, you may want students to:</p> <p>Present Hoffman's proof of the airport conjecture.</p> <p>Present a construction technique for the 120 spot and explain why it works.</p> <p>Present the proof of the airport conjecture based on Rich's function.</p>	
6.18	Problem 1: This problem addresses various topics related to proof; students pick one or more to answer.	
6.19	<p><i>Write and Reflect</i> Problem 5: Write about the role of proof.</p> <p><i>Checkpoint</i> Problem 18: Summarize what the four previous problems have shown.</p>	<p><i>Write and Reflect</i> Problem 9</p> <p>Have students answer the question posed after "The Plan": How do the three steps imply that a circle encloses the most area for a given perimeter?</p>
6.20	<p>Problem 1: State the problem Princess Dido solved, understanding the story and connecting it to the mathematics students have been learning.</p> <p>Problem 2: Explain Aristotle's statement about circular wounds, connecting the reading to the mathematics.</p> <p>Problems 4–6: Understand the three theorems of Zenodorus and how they connect to, but do not prove, the Isoperimetric Theorem.</p>	

INTRODUCTION TO
OPTIMIZATION

OVERVIEW

This section of the module introduces the notion of optimization and provides practice using geometric techniques to solve maxima and minima problems. There are lots of problems here, and your students don't need to solve *all* of them. Key problems are pointed out in the notes for each investigation. We also note particularly challenging problems.

There are several “framing problems” in this section that are revisited in later investigations and are solved using more than one technique. These problems help students to see connections between the various methods they are learning.

This is mainly a reading activity with a couple of discussion problems to introduce the meaning of *optimization*. It also gives students an idea of what represents an optimal solution and of the kinds of problems that call for one.

The main ideas in this investigation are:

- Understanding what is meant by the phrase “optimization problems”;
- Understanding how to solve some numerical optimization problems through trial and error.

One discussion that might weave its way throughout this module, beginning with this investigation, is the difference between finding a maximum or minimum value experimentally and finding that value through theoretical analysis. For example, see the sidenote for the solution to Problem 20 of Investigation 6.2 in the Solution Resource and/or the solution to Problem 9 of Investigation 6.19.

No specific prerequisites are needed here. The only preparation required is to decide ahead of time whether you want to use something like a spreadsheet to model the food pantry and money conversion problems.

You might also want to use tables, graphs, or graphing calculators to model some of the sample optimization problems listed on the first page of the investigation. Another possibility is to bring a map to look at a shortest route for driving from New York to Chicago. None of these examples *must* be explored; they are provided in case you wish to use them.

Materials: Calculators are invaluable for solving the two problems in this investigation if students don't use a spreadsheet.

Technology: The problems in this investigation are ideal for spreadsheet software, but it is not essential.

The night before: You might want to have students bring in the current exchange rates from the most recent Sunday travel or business section of the newspaper.

TEACHING THE INVESTIGATION

This investigation should only take 1 or 2 days of class time. If your students don't read the investigation the night before, you might start the first class by asking them to read the first page (perhaps even out loud). Then hold the suggested discussion about optimization problems.

The two problems can be solved individually or in groups, but students will likely need some guidance to know when they're "done," that is, when they've found the optimal solution. This would be a fruitful topic for a closing discussion or for a final write-up by students.

In the peanut butter problem (Problem 1), students most likely will use an experiment to check all possible reasonable cases. This approach can *combine* experiment with analysis: analysis to limit the search to reasonable cases, and exhaustive experiment to pick the right one. Guess-and-check is a reasonable method when there are only a finite number of possible answers.

In the exchange-rate problem (Problem 2), students might set up and solve an equation to find the break-even point: $1.55 \times p = (1.65 \times p) - 2$. It is clear from a little experimenting that, if you have less than p pounds, you should go to the currency service, and if you have more than p pounds, you should go to the bank.

Again, the difference between this algebraic method and an experimental approach using a spreadsheet is important, and should be emphasized in class discussions.

ASSESSMENT AND HOMEWORK IDEAS

- This investigation could be assigned as reading homework. You might have students try to solve the two problems on their own at home. Then spend some time in class looking at the problems after a discussion on the nature of an optimization problem.
- Alternatively, if you do the investigation in class, you might assign write-ups of the two problems, along with a reading of the first page of the next investigation, for homework.
- A closing writing topic could be "How can you be sure you've found the best solution to the peanut butter problem?"

USING TECHNOLOGY

These are some Logo programming ideas for Problem 1.

Programming Idea 1:

Here the amount of peanut butter to fill **large** jars, the amount to fill **small** jars, the total amount **packed**, and the amount **remaining** are all separate functions.

```

to FillLarge :lnum      to FillSmall :snum
  output :lnum * 32      output :snum * 9.5
end                      end

to Pack :lnum :snum
  output (filllarge :lnum) + (fillsmall :snum)
end

to Remainder :lnum :snum
  output 272 - pack :lnum :snum
end

```

Type **print filllarge 8** or **print fillsmall 4** to find out how much peanut butter can be packed into 8 large jars and, independently, how much peanut butter will fit into 4 of the small jars. Alternatively, type **print pack 8 4** to get the total amount packed into 8 large and 4 small jars. Type **print remainder 8 4** to see the amount remaining unpacked of the original 272 ounces.

Programming Idea 2:

The Logo procedure can be translated directly from the algebraic expression for the problem: $r(l, s) = 272 - (32l + 9.5s)$.

```

to Try :lnum :snum
  output 272 - (:lnum * 32 + :snum * 9.5)
end

```

Type **print try 8 4** to see the value of $R(8, 4)$, and then try other values to find the lowest.

Programming Idea 3:

This is an embellishment that will appeal to some students' sense of elegance (and desire to program), but goes beyond what is necessary for solving the problem. Using this approach, students can specify a number of, say, large jars only, and let Logo tell them how many small jars can be filled with the remaining peanut butter as well as how much peanut butter is left unpacked. There are many ways of doing this. The program suggested below makes use of all the procedures from Idea 1 and adds some new ones.

```
to AfterLarge :lnum
  output 272 - (fillarge :lnum)
end
```

```
to HowManySmall :pbutter
  output int (:pbutter / 9.5)
end
```

```
To SmallWLarge :lnum
  output HowManySmall AfterLarge :lnum
end
```

```
to FigureItOut :lnum
  print (sentence [If I fill] :lnum [large jars,])
  print (sentence [then I can fill] SmallWLarge :lnum [small jars,])
  print (sentence [and have] remainder :lnum SmallWLarge :lnum
    [ounces left over.])
end
```

AfterLarge tells how much is left over after filling the **lnum** large jars. **HowManySmall** tells how many small jars can be filled (an integral number) with a given amount of (remaining) peanut butter. **SmallWLarge** puts these together to calculate how many small jars can be filled given a certain number of filled large jars. Typing `print SmallWLarge 5`, for example, should result in the computer typing 11 as the number of small jars filled. **FigureItOut 6** does its own printing (no need to type “print FigureItOut 6”), and the result is:

```
If I fill 6 large jars,
then I can fill 8 small jars,
and have 4 ounces left over.
```

Investigation
6.2

Student Pages 3–19

MAKING THE LEAST OF A
SITUATION

Materials:

- balls for the discussion (The Léonárt Sphere™ is ideal; see “Additional Resources”)
- string and nails models (optional; see “Without Technology”)

Technology: One way to experiment with the various situations is to build dynagraphs in your geometry software to model the changing lengths. See “Using Technology”.

You might point out to students that the word “distance” is not defined so rigorously in everyday life. When we talk about the distance from Boston to New York, we usually mean the length of the shortest path that we can travel by car between the two cities. There may not be a road which follows the *actual* shortest path.

OVERVIEW

Most of these problems ask students to find methods to minimize paths: from point to point, from point to line, or along a path given certain constraints. The outcomes of these problems, to be used frequently in later problems, should articulate of some basic minimizing principles in geometry:

- The shortest path from a point to a line is along the perpendicular from the point to the line.
- The shortest path between two points is along the segment between them.
- Every point on a perpendicular bisector is equidistant from the two endpoints.
- Reflecting a point over a line forms a segment for which the line is a perpendicular bisector.

Each of the problems can be investigated experimentally, with the geometric content coming from the investigation. There is no prerequisite knowledge, but students will encounter the terms perpendicular and perpendicular bisector, and they will need to reflect a point over a line.

The reflection method will be used frequently in later problems.

TEACHING THE INVESTIGATION

You can ask students to read the first page of the investigation and do Problems 1 and 2 for homework the night before you begin the investigation.

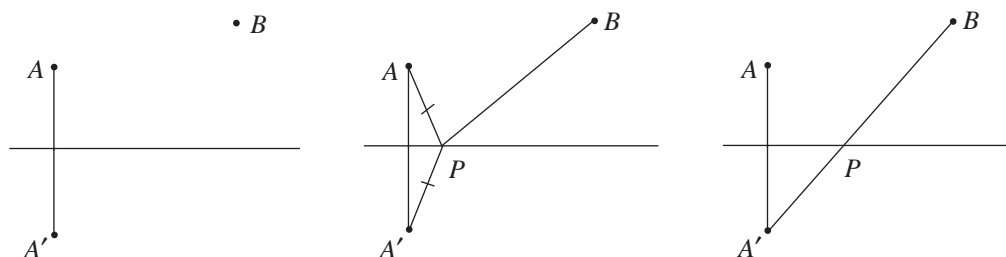
The opening class discussion is optional. You may want the students to discuss the questions in small groups and then hold a short whole-class “reporting session.” That way, if you have balls available, more students will get the opportunity to play with the notion of shortest path on a sphere before hearing the answer.

After students work on Problems 1 and 2, there should be a short discussion in which ideas about shortest distance between two points and shortest distance from a point to a line are clarified. Likewise, after working on Problems 3–6 the “reflection principle” should be discussed. Problems 7–9 can be done in class or for homework; they ask students to apply the reflection principle in other contexts.

What's coming up? The reflection method will be used frequently in later problems, which provide a key insight for one of the proofs in the section "The Most Area."

The outcome of Problems 3–4 is the “reflection principle:”

I can take A and reflect it over the line to make A' (so that the line is the perpendicular bisector of $\overline{AA'}$): Then, going from A to the line to B is the same as going from A' to the line to B , because $AP = A'P$ for any point P on the line. So, just connect A' to B and that's where you should put P .



Getting to a conjecture in Problem 5 depends on students asking themselves the right questions. If they only try the two examples given in the Student Module, urge them to experiment with others, such as A being $\frac{1}{3}$, $\frac{1}{4}$, $\frac{3}{4}$ as far from the riverbank as B .

Problem 13b introduces the idea that the minimum value of a system sometimes occurs at a boundary point. This is the “edges of the domain” situation that calculus students learn about. A theorem studied in calculus states that a function that is continuous and differentiable on a closed interval always assumes its maximum and minimum values either at the endpoints of the interval or at the points where the derivative is zero. In other words, the extreme values of a function occur at special places or at the edges of the domain.

Problems 20 and 21 can be used to apply or develop the classical formulas that connect sizes of angles and the corresponding sizes of their intercepted arcs. (If P is outside the circle, the size of $\angle SPT$ is smaller than if P is on the circle.)

ASSESSMENT AND HOMEWORK IDEAS

- Problems 8 and 9 will make good assessments of whether the students see the related geometry within these slightly-different contexts.
- The “Checkpoint” problems ask students to state explicitly (in their own words) the minimizing principles mentioned above. These would make a good final homework or quiz.

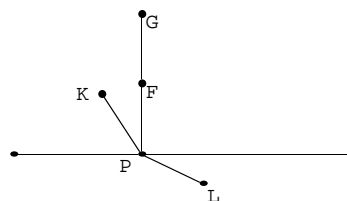
- “Take it Further” Problems 20, 24, and 26–28 are particularly good problems that are related to earlier problems, but with a twist.
- Problems 3–6 and 16 from Investigation 6.5 could all be done after this investigation is completed.

WITHOUT TECHNOLOGY

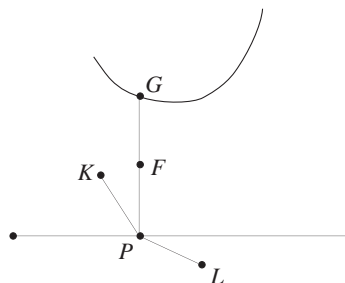
To model the situation for Problem 2 with string and nails, use two nails and a stationary piece of string to represent the shoreline. Include a washer on the string to represent the point on the shoreline; loop a second piece of string from point L through the washer to point K to represent the varying possible paths over land and in the water. The washer could be alternately fixed at several places with a thumbtack, and the length of the string needed to reach point K marked with different colored markers for each place fixed with the thumbtack.

USING TECHNOLOGY

If you model the situation for Problem 2 in a geometry software environment, you can drag P back and forth and observe how the total trip distance changes. To do this, you might want to measure $KP + LP$. Another way to do this is to build a *dynagraph* of the function $P \mapsto KP + LP$. For example, we erected a perpendicular stick up from P and found points F and G so that $PK = PF$ and $PL = FG$.



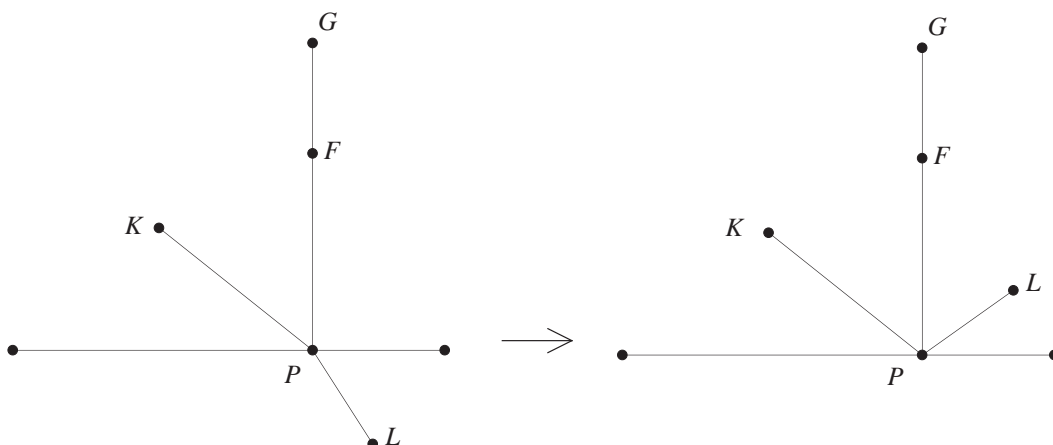
So, the total length of \overline{PG} is the length of the trip. As you drag P back and forth, you can observe the height of G , and the minimum trip will be at the spot where G is lowest. If you put a *trace* on G , then dragging P back and forth produces a graph of the function $P \mapsto KP + LP$:



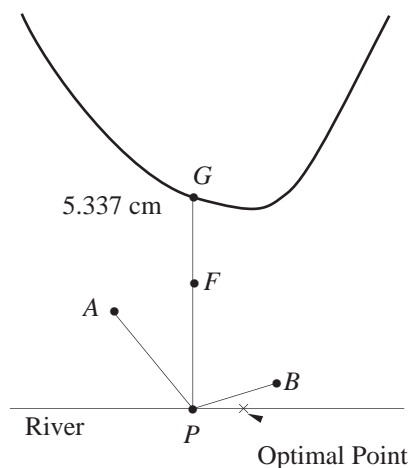
You can verify that the minimum point on the dynagraph occurs when K , P , and L are collinear. What happens if you put a trace on F ? See the “Mathematics Connections” section for further scenarios applicable to this problem.

You might want to give this technique a name (the “reflection principle,” for example). It’s useful in solving Problems 7 and 9.

Problems 3 and 8: If students built a dynagraph for Problem 2c, you could ask them to think about some way they might alter that setup to represent Problem 3 or Problem 8. By taking the “inland” point (it was called L in that problem) and simply flipping it (by using the “reflect” command) over the shore into the water, they’ve got the setup they need. (This action might also provide a key insight for making the theoretical analysis of the problem.)

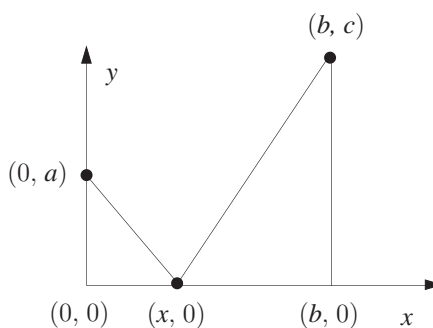


Relabeling and running the dynagraph, we get a picture like this:



MATHEMATICS CONNECTIONS

Problems 3 and 8: The graph of the function can, of course, be generated without geometry software. Students who have had some coordinate geometry and the requisite algebra can write the analytic description of the distance as a function of P 's position, starting with a diagram like this:



The distance from $(0, a)$ to $(x, 0)$ is $\sqrt{a^2 + x^2}$, and the distance from $(x, 0)$ to (b, c) is $\sqrt{(b-x)^2 + c^2}$. Thus, the total distance $D(x)$ is $\sqrt{a^2 + x^2} + \sqrt{(b-x)^2 + c^2}$.

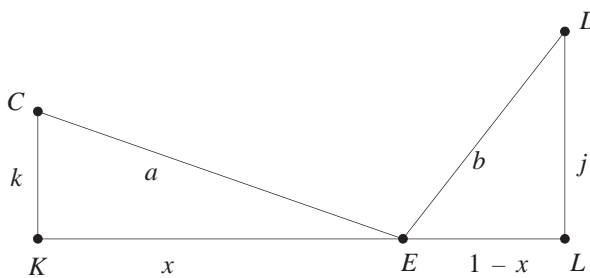
Sometimes people who have physical handicaps that don't allow them to run very fast can swim very fast instead.

Maybe you could ask a science teacher to come into your class and discuss Snell's Law.

(In France, Snell's Law is known as "Descartes' Law.")

You might want to discuss other optimization scenarios relevant to Problem 2 or Problems 3 and 8. For Problem 2, for example, suppose you can run faster than you can swim. Then any line segments through the water have a different "cost" (or "weight," or "angle") than do line segments over the land. An analogous situation comes up in optics, where a light beam traveling in air "bends" (its direction is *refracted*) when it enters water. The description of this phenomenon is given in Snell's Law.

Field-tester/teacher Jane Gorman posed the burning tent problem to her class. Several students observed that if you want to get to the tent in as short a *time* as possible (time is crucial if your tent's on fire), then you might want to take into account the fact that you can probably run faster with an empty bucket. Where should you hit the river in this case? It turns out that if you can run r times as fast empty, then the best place to land is at the spot where the cosines of the angles of incidence and reflection have ratio r . This problem will come up later (see Problem 38 of Investigation 6.4), where students will use contour lines to obtain a qualitative solution. But we thought you'd enjoy seeing a simple solution that uses calculus and that generalizes the "unweighted" case nicely:



If your rates are r_1 empty and $\frac{r_1}{r}$ full, then the time it takes to get to the tent is

$$\frac{a}{r_1} + \frac{rb}{r_1}.$$

Suppose the lengths are as given in the figure (there's no loss of generality in assuming that $KL = 1$). Notice that k and j are constants, but a and b depend on x . Suppose that you can run $\frac{1}{r}$ times as fast from E to D (the tent) as you can from C to E . So, r might be 2 or 3, or any number greater than (or equal to) 1. Then if $s = a + rb$, s is a positive constant times the total time taken to get to the tent, so we want to minimize s .

Since everything depends on x ,

$$\frac{ds}{dx} = \frac{da}{dx} + r \frac{db}{dx},$$

and, since

$$a^2 - k^2 = x^2$$

and

$$b^2 - j^2 = (1 - x)^2,$$

we have

$$2a \frac{da}{dx} = 2x$$

and

$$2b \frac{db}{dx} = 2(x - 1),$$

so

$$\frac{da}{dx} = \frac{x}{a}$$

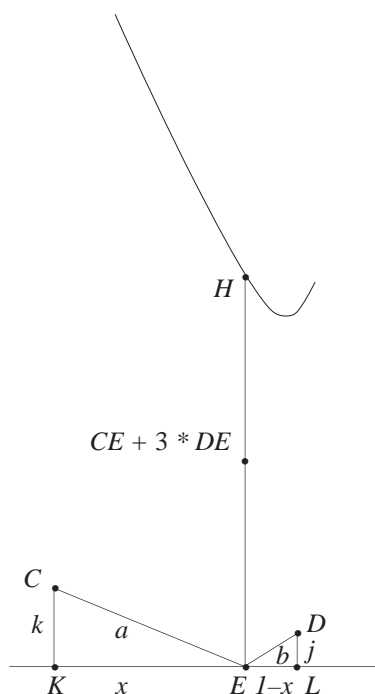
and

$$\frac{db}{dx} = \frac{x - 1}{b}.$$

Substituting, we have

$$\frac{ds}{dx} = \frac{x}{a} + r \frac{x - 1}{b}.$$

But a dynagraph experiment shows that s is one of those functions that has a minimum when its derivative is 0:



so we have

$$0 = \frac{x}{a} + r \left(\frac{x-1}{b} \right),$$

or

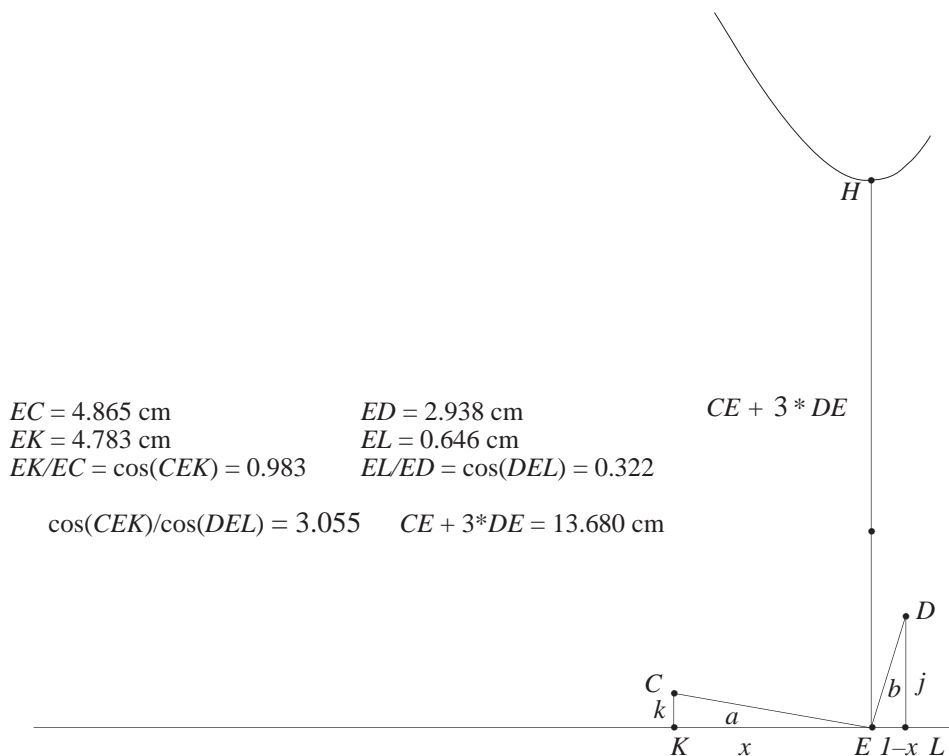
$$\frac{x}{a} = r \left(\frac{1-x}{b} \right).$$

But, in terms of trigonometry, this just says that

$$\cos \angle CEK = r \cos \angle DEL.$$

For example, if you can run with a full bucket $\frac{1}{3}$ times as fast as you can run with an empty one, you should land at the spot where

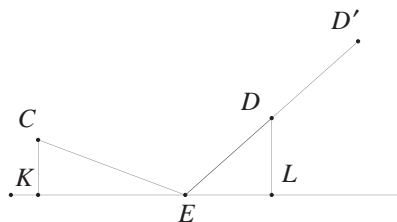
$$\cos \angle CEK = 3 \cos \angle DEL.$$



Notice that if $r = 1$ (as we intended in the original burning tent problem), we can use the fact that the angle of incidence wants to be the same as the angle of reflection.

This experiment requires geometry software.

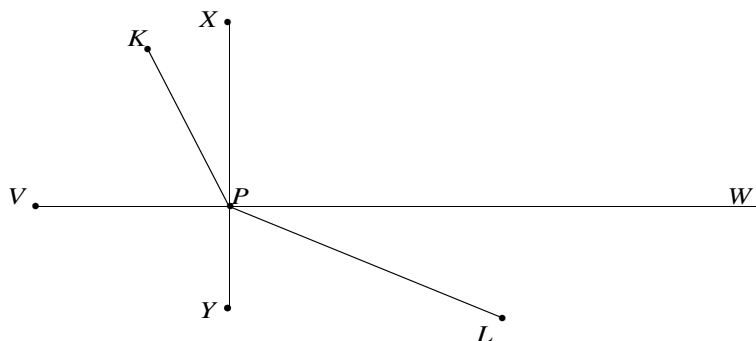
If this problem comes up in your class discussions before students get to contour lines, you might suggest (if they don't) that if you run half as fast on the last leg of the trip, you really want to minimize the sum of the distances from you to the river and *twice* the distance from the river to the tent. This insight could be part of a nice digression into rate, time, and distance, and it also means that the problem is open to experiment with geometry software.



Minimize $CE + ED'$ as a function of E .

It comes as a surprise to many students that the best spot isn't simply the one that minimizes the amount of bucket carrying (that would be at the foot of the perpendicular from the tent to the river), but it's just a bit off from this position. Wrestling with this counterintuitive fact could be very productive. In any case, you'll want to help students make as much progress as they can, stating as many conclusions as possible, without saying too early that the complete solution to the problem requires more mathematics than they have right now.

By the way, this same solution applies to the "weighted" version of the run-and-swim problem (Problem 2 of Investigation 6.2).



If you can run r times as fast as you can swim, then you should land at a place where

$$\cos \angle WPL = r \cos \angle KP V.$$

This is analagous to the situation where a light beam travels from L through water at P , is *refracted*, and then travels on to K . In optics, Snell's law says that

$$\frac{\sin \angle YPL}{\sin \angle KPX} = r,$$

where r is the “index of refraction.” Is this equivalent to our formulation?

ADDITIONAL RESOURCES

The Léonárt Sphere™ is available from

Key Curriculum Press
P.O. Box 2304
Berkeley, CA 94702

Investigation
6.3

Student Pages 20–27

MAKING THE MOST OF A
SITUATION

Materials:

- graph paper
- string and nails models for ellipses
- scratch paper and scissors

Technology: Geometry software is helpful for Problems 1–3 and 8–9.

If students have worked through the module *The Cutting Edge*, that work will help tremendously with these problems.

See the “Assessment and Homework Ideas” below for another way to work through Problem 4 with your class.

OVERVIEW

This investigation includes several problems related to the question presented in the class discussion on page 20: “For all shapes with the same perimeter, which one has the greatest area?” This is known as the *Isoperimetric Problem*, and dates back to antiquity. The main result is that the circle maximizes area for a given perimeter. The proof of this will be presented in the section “The Most Area”.

Main ideas that students should get from this investigation:

- For all rectangles of fixed perimeter, the square has the greatest area.
- In general, for a polygon with a given number of sides and fixed perimeter, the regular shape encloses the greatest area.
- For fixed perimeter and regular polygons, increasing the number of sides increases area.
- Properties of angles inscribed in circles

Students need to know some general notions about area, as well as some area formulas (triangles and rectangles, and methods for approximating other regular polygons). Some algebraic manipulations (using variables in perimeter and area calculations) are necessary as well.

TEACHING THE INVESTIGATION

After the opening discussion, students could work on Problems 1–3 in class. They will need either geometry software or graph paper to compare areas of various figures.

Problem 4 is important and deserves a lot of attention. You may want to have students read over the proof for homework. In class, you can ask students to present the argument orally (or do so yourself). If they have not worked on cutting problems before, they should actually make the rectangles, cut them as shown, and rearrange the pieces. Students can work on the three parts of the problem in class and for homework, and the next day they can present their solutions for part c. Problem 5 generalizes Problem 4. Students may need some guidance with the algebra, but the point is to understand the cutting argument.

Problems 8 and 9 involve a different kind of optimization (maximizing an angle rather than an area). Before going on to this topic, you may want to do the relevant “Checkpoint” problems to assess the area maximizing techniques.

Problems 1–3 are closely related, and the similarities should be made clear to students. If they can solve Problem 1, they can solve Problem 2 by thinking about parallelograms as two congruent triangles joined at one of their identical sides (the diagonal of the parallelogram).

Doing these problems sets the stage for students’ work on the isoperimetric problem in the last section of the module.

In Problem 18, your students may end up faced with comparing the areas of three polygons: a square with sidelength 210, a regular pentagon with sidelength 168, and a regular hexagon of sidelength 140. You can ask them to *approximate* the areas of each of these figures. They are different enough (rough calculations show areas of about 44,000, 49,000, and 51,000, respectively) so that students can see that the hexagon wins. The situation is similar for Problem 19.

ASSESSMENT AND HOMEWORK IDEAS.....

Give each group the dimensions of their rectangle in an envelope. This adds to the drama.

- Problem 4 is a good assessment problem to have students present to the class. Have the class work in groups, assign each group a particular rectangle of perimeter 128 (30×34 , 28×36 , and so on), and have each group present their proof. This will get tedious, but that’s the point. Ask the class to describe the common features of *all* the proofs, and use these features to construct a generic argument that will work for all rectangles of perimeter 128.
- Once students have solved Problems 8 and 9, an interesting follow-up question would be: Can you *construct* the desired circle (either in your geometry software or with hand construction tools)? “Checkpoint” Problems 12 and 13 are good follow-up questions for these ideas.
- “Checkpoint” Problem 11 is a good way to see if students have understood the use of a “cut-up-and-rearrange” argument.
- “Take It Further” Problem 24 is a good way to get students thinking about what they know. Why does it make sense for the answer to this question to also be a square?
- From Investigation 6.5, Problems 8–10 could be done after this investigation is completed. (Problem 10 asks students to think about surface area/volume relationships instead of perimeter/area relationships.)

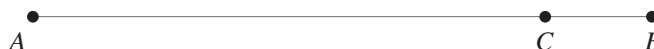
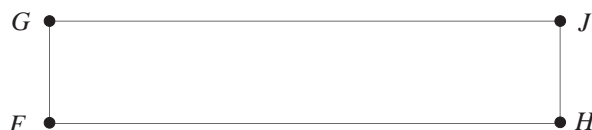
USING TECHNOLOGY

Here is a way to approach Problem 3 with geometry software.

$$\text{Area} = 8.832 \text{ cm}^2$$

$$FH = 6.696 \text{ cm}$$

$$FG = 1.319 \text{ cm}$$

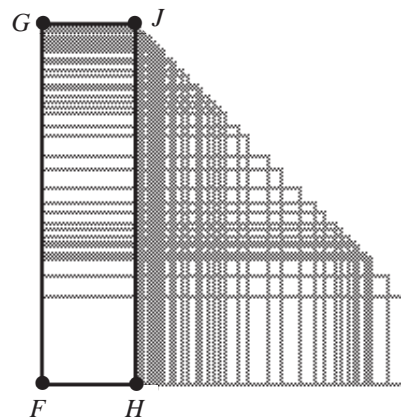


The figure above is a scale drawing of the house, in which AB is half the perimeter (its measure is 8 cm in the sketch). C is a movable point on \overline{AB} . The rectangle is constructed to remain a rectangle, so $FH = AC$ and $FG = CB$. When you measure FH and FG and calculate their product, the values can be displayed on the screen. As C slides back and forth across \overline{AB} , the rectangle changes—its area changes, but its perimeter stays at 16 cm:

$$\text{Area} = 10.501 \text{ square cm}$$

$$FH = 1.649 \text{ cm}$$

$$FG = 6.366 \text{ cm}$$



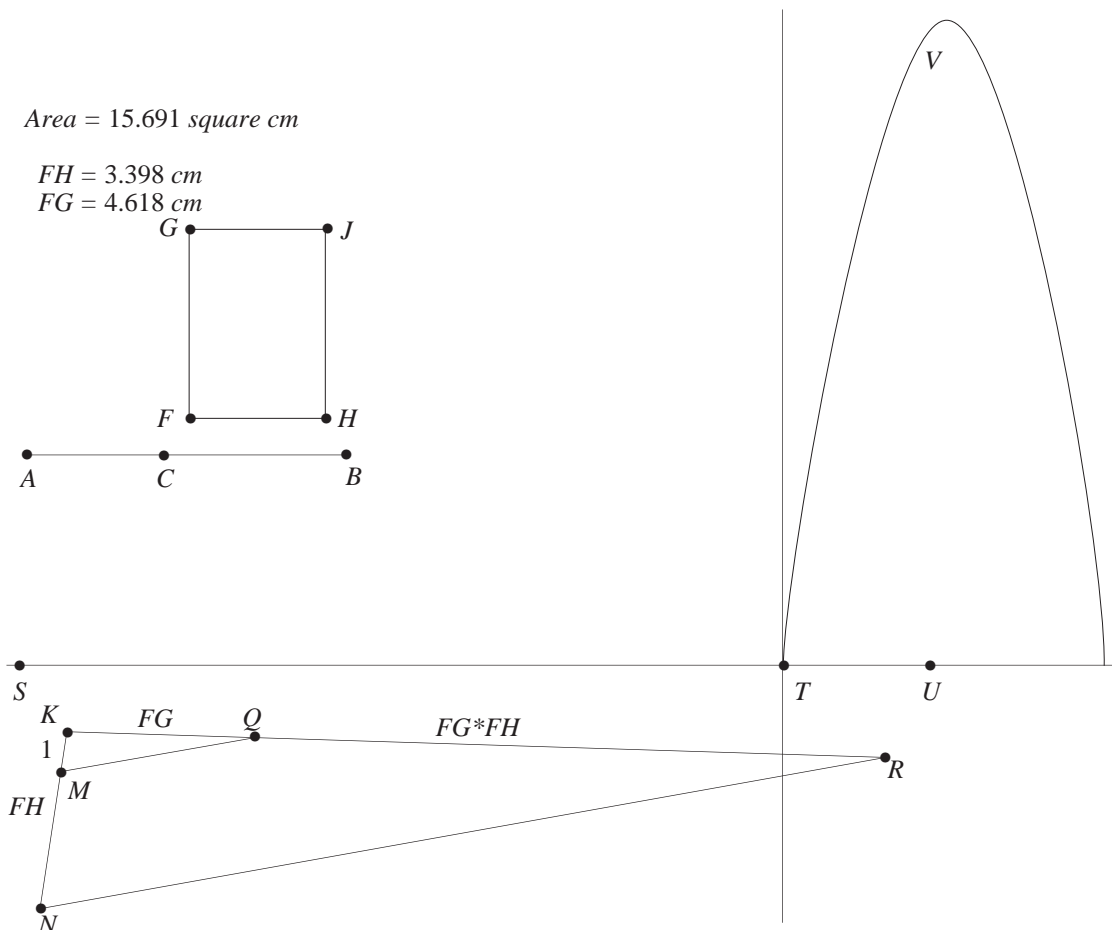
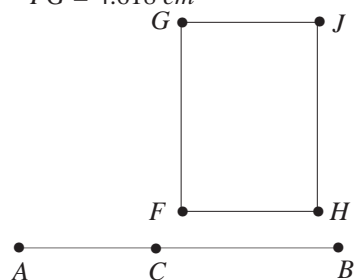
On one axis, copy a length equal to FH . On the other, copy a length equal to $FG * FH$ ($FG \cdot FH$).

A more elaborate way to visualize the situation is to make a dynagraph of the situation as in the next figure.

Area = 15.691 square cm

$FH = 3.398$ cm

$FG = 4.618$ cm



Depending on your software, you may have to construct the length $FG \times FH$ using the Euclidean method for multiplying lengths.

ADDITIONAL RESOURCES

An entire book has been written about ideas associated with the arithmetic-geometric mean inequality, *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity* (John Wiley, 1987).

CONTOUR LINES AND
CONTOUR PLOTS

Materials: String-and-nails models (described in the Student Module) provide a physical representation of contour lines.

Technology: Geometry software is extremely helpful in this investigation and is assumed for Problems 17 and 34. The investigation can be done without technology by using physical models and skipping or altering those problems.

OVERVIEW

The idea of function is central to this investigation. Students will explore functions on the plane ($\mathbb{R}^2 \rightarrow \mathbb{R}$) and use contour lines to find minimal values for these functions along a path. The investigation has 45 problems; your students don't need to do all of them. Key problems are pointed out in "Teaching the Investigation" below. Key ideas:

- Contour lines represent all the places with the same value for a function ("equally good/equally bad places").
- The minimal value for the function along a path is at its tangent point to the contour line with the smallest value.
- Circles are contour lines of constant distance from one point.
- Ellipses are contour lines of constant distance from two points.
- Arcs of circles are contour lines of constant angle.
- Surface plots are three-dimensional plots of contour lines. They show the value of the function along a contour line by their distance from the x - y plane.

Several problems from Investigation 6.2 are revisited here. This gives students another way to look at minimization problems, so it is important that they've done the earlier work. Vocabulary and ideas about functions are introduced here, particularly in the "Ways to Think About It" reading, so your students do not need to know about them previously.

TEACHING THE INVESTIGATION

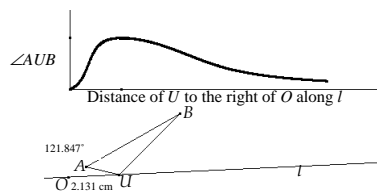
You can have students read the first page and answer Problems 1 and/or 2 as homework the night before you begin the investigation.

If students have not read the beginning of the investigation, you may ask them to read it aloud in class so that they can ask questions as they go along. Problems 3 and 4 are important for the rest of the investigation and need some focused class time. If students don't understand the answers, they will have difficulty with the rest of the investigation. Problem 5 is a good follow-up/assessment to see if students can reinterpret the map in a new context.

Looking for patterns or invariants: Contour lines are invariants. They all have the same value for a given function. If your class uses the language of invariants, you should use it here to make that connection.

What's coming up? Problems 17 and 20 reintroduce ellipses as contour lines of fixed distance from two points, using the language of function. Problem 22 extends this to the idea of fixed distance from three fixed points. If you plan to do the extended investigation on “The Airport Problem,” you should be sure to preview it with the problems here.

When the running path is not parallel to the posts, the graph of the angle change is not symmetric. Here is a picture (generated in The Geometer's Sketchpad®) of the change.



Problems 6–9 are really all one problem and would be good for homework with a brief in-class discussion afterwards to be sure that students understand that a circle is the set of points a constant distance from a fixed point (this is familiar to most high school students). Problems 10–15 focus on arcs of circles as curves of constant angle. This is less familiar to most students, and the early problems may be difficult. These should be done in class so that you can provide guidance and hints. Marking points of constant angle with the model described in “Without Technology” or with software may provide key insights.

The “Ways to Think About It” reading introduces the concept of function and ties it to what students have been doing. It should be read carefully and discussed. Problem 18 is one they have already solved (the circle as constant distance from a point), but phrased in the language of functions. It is important for students to make the connection between this problem and Problems 6–9.

“Revisiting the Burning Tent” is a good section for tying together minimization strategies your students learned in Investigation 6.2 and working with contour lines. If you want to shorten this section, and your students already know a bit about ellipses, you can just do Problems 31–33 or 34 with your class.

In a full investigation of Problem 14, students will find the maximum value for the kicking angle at the point of tangency between the player’s running line and a contour line. The notion of tangency of contour lines maximizing (or minimizing) some value is revisited often in mathematics. The horizontal tangent to the curve that students study in calculus is an example of this idea. Horizontal lines are “contour lines” along which the value of y does not change as x changes. Where a horizontal line is tangent to the graph of $y = f(x)$, we see local (or global) maxima or minima of $f(x)$. Here we see circles performing the same function. Circles are the contour lines along which the value of $\angle AUB$ does not change as U changes. Where a circle is tangent to some path that U might take, $\angle AUB$ is at its maximum.

This idea reappears in Problem 17, where the contour lines will be ellipses. In the escape-from-the-crazy-pool problem, the level curves were concentric circles with the swimmer’s original position at their center. All points along any circle are equally distant from the swimmer. The smallest circle that is just tangent to the pool determines the direction by its point of tangency. Larger circles that are tangent to other parts of the pool show local minima—swimming to their point of tangency is shorter than swimming to other points near that point of tangency—but are not the global minimum identified by the smallest tangent circle.

In Problem 23, the fact that points A and B cannot be included on the contour plot may be missed by many students.

ASSESSMENT AND HOMEWORK IDEAS.....

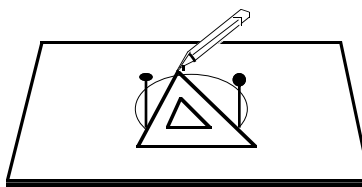
- If you cover Problems 10, 11, and/or 12 in class, any combination of Problems 12–14 would be a good homework assignment or assessment.
- Problems 36 and 41 are good follow-up assessment problems to the “Revisiting the Burning Tent” section.
- Problem 45 is a relatively-easy problem that asks students to extend the notion of contour lines to three dimensions. It would be a good check of both their concept of contour lines and their visualization.
- From Investigation 6.5, Problem 7 (ask students to solve it using contour lines) and Problem 11 are good choices.

Any problems from Investigation 6.5 could be done now; some will function as reviews of earlier investigations.

WITHOUT TECHNOLOGY

You do not need to build all of the string-and-nails models yourself. Your students can do that during class one day. Even the stiff angles described below can be made by students in class using stiff cardboard, protractors, and good scissors.

Students can also perform the experiment in Problems 10–14 with physical manipulatives, using a stiff “angle” (a plastic triangle or a carpenter’s L), and two nails in wood. Slide the “angle” around with its sides against the nails. The vertex traces out a circular arc.



USING TECHNOLOGY

For Problems 10–14, students may use geometry software to find the points experimentally. They can set up two points as goalposts and a third point as the player. Measuring the appropriate angle, they can move the player to some location at which the angle is 40° and mark that position softly on their screen (or a transparency taped over the screen) with an erasable marker. After they have found and marked enough such points, they may be ready to make a conjecture. If everybody uses goal posts the same distance apart, their transparencies may be superimposed to combine their data.

Alternatively, students may use Trace Locus to leave a trace of the player's position as they move it while watching vigilantly to keep the measured angle between, say, 39.9° and 40.1° . Even with this technique to capture the data, students may want to transfer their results (with a felt tip marker) to a transparency or tracing paper.

By varying the position of the player while maintaining a constant angle, you are generating the arc in which all those angles are inscribed.

DRAWING CURVES WITH GEOMETRY SOFTWARE

This section is quite long. You may want to come back to it when you need it, or just skip it if you're already familiar with these software packages.

In this investigation and for the rest of the module, students will be asked to construct contour lines for various functions $f: \mathbb{R}^2 \rightarrow \mathbb{R}$. We encourage you and your students to devise mechanical devices for constructing such curves (the famous pin-and-string construction for an ellipse is such an example).

Powerful media like Mathematica®, which have built-in facility for drawing contour plots, may or may not be available to you. Essentially, there is a primitive (it's called **ContourPlot** in Mathematica) that takes a function defined by you (usually in algebraic notation) and several other parameters, and then produces a contour plot for the function.

For example, if you have four colors, color the pixel red if the value of the function is less than 3, blue if it's greater than or equal to 3 and less than 10, yellow if it's greater than or equal to 10 and less than 100, and black if it's greater than or equal to 100. You end up with “contour bands” and then you can go back to adjust the intervals to get better pictures. The boundaries between bands are contour lines.

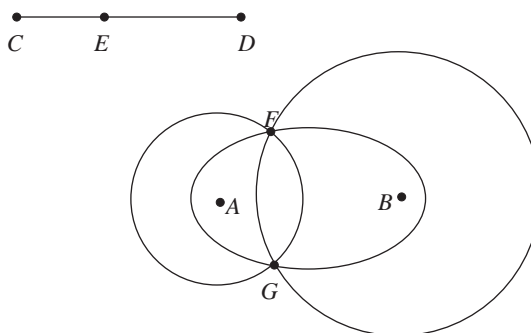
Setting up the constraints involves functional thinking in the sense of constructing algorithms. Manipulating the sketch involves functional thinking in the guise of reasoning by continuity. Gathering up the points in the path into an object that itself depends on parameters in the sketch is a good example of abstraction.

You can write your own program to do this in Logo. One easy way is to instruct the computer to scan the screen, pixel by pixel, and at each data point, to evaluate the function at hand on the coordinates of the pixel, and color it accordingly. This tedious process (tedious for the computer, not for you) pieces together local information about the function and produces a picture of the global behavior of the function.

Another method, one that really builds habits of algorithmic thinking, performing thought experiments, reasoning by continuity, and visualization, is to use geometry software to generate contour lines. This can't always be done, and the capabilities will vary from system to system, but simply being able to answer the question “Can I figure out how to make the software generate the contour lines for this function?” is a valuable skill for students to develop. In the rest of this section, we give some methods for generating curves described by various geometric constraints, using common features of many geometry software environments.

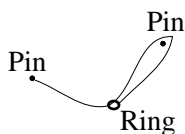
The idea is to bring back the old notion of “locus of a point”: if a point moves in such a way that it satisfies certain constraints, what path does it trace out? Geometry software environments allow students to set up the constraints and then to trace the path of a point that is subject to them. This “tracing point” moves as a result of the direct manipulation of one or more features of a sketch; in other words, it is a *function* of these features. When students construct such sketches, they are constructing computational models for functions, and the behavior of such functions can be *experienced* in a very kinesthetic way when students experiment with their creations. In addition, some geometry software allows students to gather up the image points of such functions into a *set* that can be manipulated by other functions.

Let's start with the typical construction of an ellipse. Given a positive constant k and two points A and B , the ellipse with foci A and B and major axis k is the set of points P so that $PA + PB = k$. Draw a segment \overline{CD} with length k , and place a “slider point” E on \overline{CD} that breaks the segment into two parts. The sum of the lengths of these two parts is always k .



$$PA + PB = CD$$

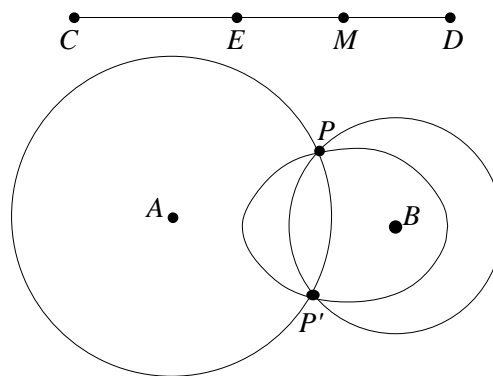
This slider point method is a computational equivalent of the pin-and-string construction of an ellipse, which is made easier with the aid of a small ring:



Tie the string to one pin, lace it through the ring, around the second pin, and tie it to the ring. Now trace the curve by putting a pencil in the ring.

Circles of radii CE and ED are constructed with centers A and B . As E is slid back and forth along \overline{CD} , points F and G (the intersections of the two circles) trace out the upper and lower halves of an ellipse with foci at A and B and whose major axis has length CD (why?).

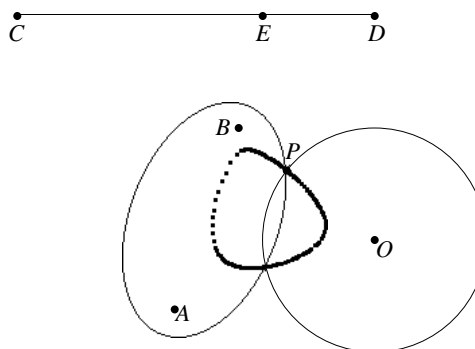
This “slider point” method is quite general. For example, rather than looking at all the points P so that $PA + PB$ is constant, students can construct functions that produce the set of points P so that $PA + 2PB$ is constant:



$$PA + 2PB = CD$$

In this sketch, points P (and P') are functions of E that are constructed in the following way: Circles of radii CE and $EM = \frac{1}{2}ED$ are drawn around A and B respectively, intersecting at P . As E is dragged along \overline{CD} , P moves in a way so that $PA + 2PB = CD$ is constant.

The slider point breaks a segment into two parts, the sum of whose lengths is constant. These parts can be used to generate figures other than circles; the set of possible loci that students can build depends only on the available primitives. In Cabri Geometry II™, for example, conic sections are built in primitives, so that loci defined as intersections of conics are possible. For example, “a generalized ellipse with three foci,” the locus of points P so that $PA + PB + PC$ is constant can be obtained from the intersection of an ellipse and a circle:



$$PA + PB + PC = RS$$

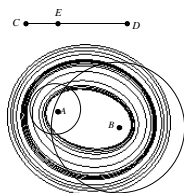
To generate the locus, slide E back and forth along \overline{RS} .

In the above, sketch \overline{CD} is divided at E . An ellipse with major axis ES is constructed on foci A and B , and a circle with radius RE is constructed with center C . P is the intersection of the ellipse and the circle.

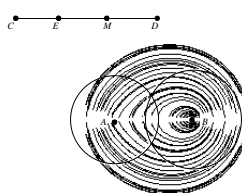
The continuous dependence of the locus on the “sliding” point E in the above examples is just the beginning. Once the locus is obtained, students can ask how *it* depends on the parameters in the sketch. In most geometry software environments, it’s possible

This encapsulation of isolated events into objects that can then be input to higher-order processes is a fundamental abstraction mechanism in mathematics.

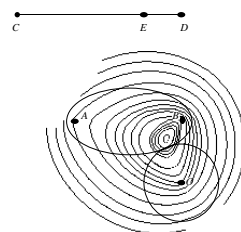
to “encapsulate” loci into objects that can be manipulated via other functions. For example, the next pictures show families of ellipses, “quasi-ellipses,” and “generalized ellipses” that grow and shrink as a function of the “length of the string”:



A family of ellipses



A family of “quasi-ellipses”



A family of “generalized ellipses”

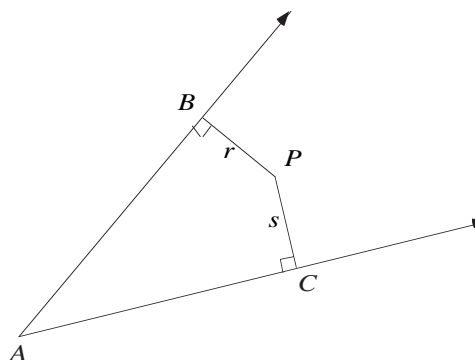
You can make a dynamic sketch of this, where k is the length of some segment in the sketch. By changing k , the lines on either side of ℓ move closer and farther away.

Assume f is defined on the interior of the angle and $f(P) = r + s$. For which point P is $f(P) = k$?

One final class of examples: In all the examples so far, we have been worrying about the distances between points. If we include distances between points and lines as part of the allowable constraint definitions, we get different kinds of contour lines.

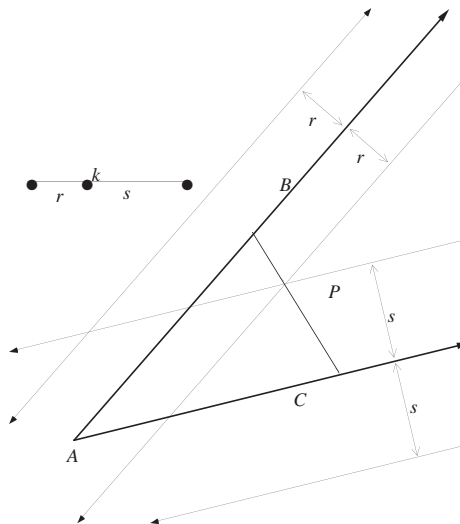
The simplest example: Suppose ℓ is a line. If f is the function that measures the distance from its input to ℓ , then the contour line for $f(P) = k$ is the union of two parallel lines, each k away from ℓ , on either side of it.

How about this one: Given $\angle BAC$ and a positive number k , what is the locus of points P in the interior of the angle so that the sum of the distances from P to the sides of the angle is k ?



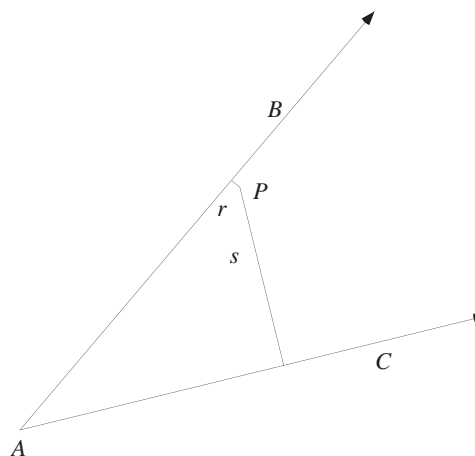
You can generate the locus by using the slider-point method. Try it: take a segment of

some fixed length k and break it into two pieces of lengths r and s with a slider point. Then make two parallels to \overrightarrow{AB} , a distance of r away, and make two parallels to \overrightarrow{AC} , a distance of k away. These four constructed lines will intersect once inside the angle. Call that point P , put a trace on P and move the slider back and forth.



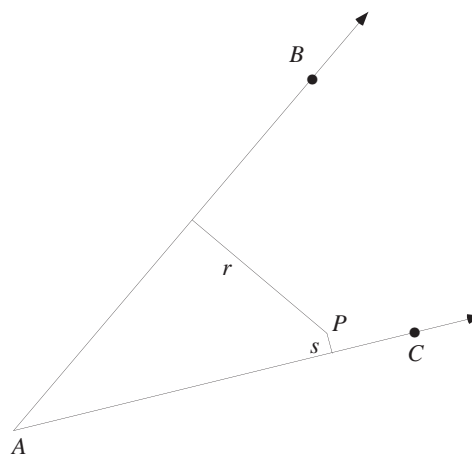
Does P trace out a line segment?

It looks like P traces out a line segment. Is that true? One way to locate some possible values of P is to imagine what happens if r or s is very small. If r is small, P is close to \overrightarrow{AB} ; in the limiting case, P is right on \overrightarrow{AB} at the spot where the distance from P to \overrightarrow{AC} is k :



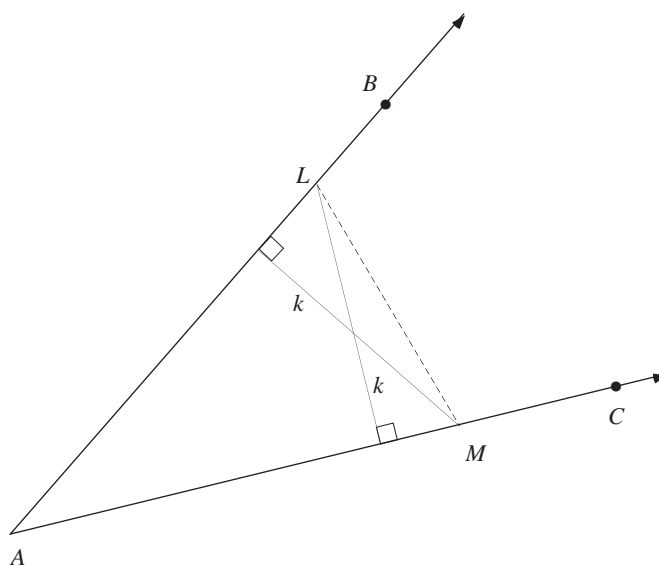
r is almost 0; s is almost k .

Another special case is at the other extreme:



s is almost 0; r is almost k .

This gives two points, L on \overrightarrow{AB} (a distance of k from \overrightarrow{AC}) and a corresponding point M on \overrightarrow{AC} (a distance of k from \overrightarrow{AB}):



Notice that $\triangle LMA$ is isosceles. From this, it's straightforward to prove that a point P in the interior of the angle is on \overline{LM} if and only if the sum of the distances from P

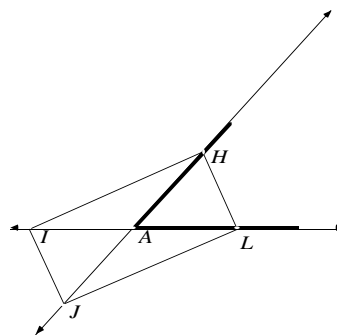
A theorem about quadrilaterals: “A quadrilateral whose diagonals are congruent and bisect each other is a rectangle.”

to the sides of the angle is k (see the solution for Problem 10 of Investigation 6.8 of the Solution Resource for inspiration).

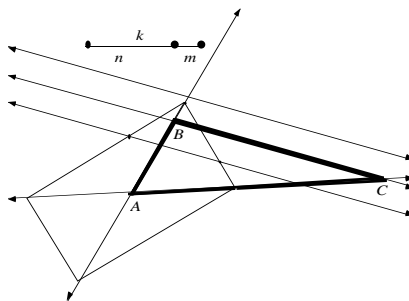
The sides of angles are rays. But suppose we extend the rays to get two intersecting lines. What is the locus of points, the sum of whose distances from two lines is constant? Work the argument above across the interiors of the four angles that are formed, use some theorems about quadrilaterals, and we have a surprising theorem:

THEOREM

The locus of points whose sum of distances to two fixed lines is constant is a rectangle.



Finally, applying the slider point method again, you can generate the locus of points the sum of whose distances from *three* fixed lines is constant.



In this picture, a segment of length k is broken into segments of length n and m with a slider point. A rectangle is constructed—the rectangle of points the sum of whose distances from \overrightarrow{AB} and \overrightarrow{AC} is n , as is the pair of parallels, each k away from \overrightarrow{BC} . This pair of parallels will intersect the rectangle at different points (depending on the position of the slider). Put a trace on each of these, move the slider, and watch what happens.

USING OPTIMIZATION

OVERVIEW

This is a set of problems designed to be used in conjunction with Investigation 6.2–6.4. They can be used for additional problems as you teach the investigations, for homework, or for a final assessment. Some of the problems here are particularly challenging and would be good for advanced classes, for projects or presentations, or for individual students who need an extra challenge.

Students should be working through, or should have completed, Investigations 6.2–6.4 when they work on these problems.

Problem 11 serves as a mechanism to help those students having trouble with some of the ideas in Investigation 6.4 (for example, the difference between lines that are tangent to and lines that cross contour lines.)

Problems 13 and 14 are particularly challenging. If students are stuck on Problem 13, ask them to solve the special case in which $\angle C$ is a right angle.

ASSESSMENT AND HOMEWORK IDEAS.....

- Each of Investigations 6.2–6.4 includes notes in the section “Assessment and Homework Ideas” about appropriate problems for use with that investigation.
- As a final assessment for the first section of the module, “Introduction to Optimization,” you might choose from the following problems: 3–9 and 11.

MATHEMATICS CONNECTIONS

Problem 10: The Solution Resource provides a proof that given a fixed volume, the cube minimizes surface area for all rectangular boxes. That proof makes use of the arithmetic-geometric mean inequality (AGM); some students may wish to see a proof of the inequality.

What’s coming up?

Problem 16 previews the extended investigation in the next section: “A Magical Mix.”

It turns out that, in order to prove the arithmetic-geometric mean inequality for three variables, it's easier to first prove it for four variables, and then use that case in the proof for three variables.

We first want to prove that, if a , b , c , and d are positive integers, then

$$\sqrt[4]{abcd} \leq \frac{a + b + c + d}{4}.$$

What we will do is make some substitutions for the variables a , b , c , and d , keeping track of what happens at each step. So we substitute the following:

$$a \rightarrow \frac{a + b}{2}$$

$$b \rightarrow \frac{a + b}{2}$$

$$c \rightarrow c$$

$$d \rightarrow d.$$

Notice that, after making these substitutions, the sum of the variables is still the same:

$$\frac{a + b}{2} + \frac{a + b}{2} + c + d = a + b + c + d.$$

The product, however, has increased or remained the same. To see this we apply the inequality in two variables to a and b and get

$$\sqrt{ab} \leq \frac{a + b}{2}$$

$$ab \leq \left(\frac{a + b}{2}\right)^2.$$

Thus, we see that

$$abcd \leq \left(\frac{a + b}{2}\right) \left(\frac{a + b}{2}\right) cd.$$

We will make a series of further substitutions, but at each step the same reasoning shows that the sum stays the same, while the product increases or remains the same.

The idea for this proof is due to the French mathematician Augustin Cauchy (1789–1857).

Next make these substitutions:

$$\begin{aligned}\frac{a+b}{2} &\rightarrow \frac{a+b}{2} \\ \frac{a+b}{2} &\rightarrow \frac{a+b}{2} \\ c &\rightarrow \frac{c+d}{2} \\ d &\rightarrow \frac{c+d}{2}.\end{aligned}$$

Then, make the substitutions:

$$\begin{aligned}\frac{a+b}{2} &\rightarrow \frac{a+b+c+d}{4} \\ \frac{a+b}{2} &\rightarrow \frac{a+b}{2} \\ \frac{c+d}{2} &\rightarrow \frac{a+b+c+d}{4} \\ \frac{c+d}{2} &\rightarrow \frac{a+b}{2}.\end{aligned}$$

Finally, substitute:

$$\begin{aligned}\frac{a+b+c+d}{4} &\rightarrow \frac{a+b+c+d}{4} \\ \frac{a+b}{2} &\rightarrow \frac{a+b+c+d}{4} \\ \frac{a+b+c+d}{4} &\rightarrow \frac{a+b+c+d}{4} \\ \frac{a+b}{2} &\rightarrow \frac{a+b+c+d}{4}.\end{aligned}$$

Denote the quantity $\frac{a+b+c+d}{4}$ by T . Since at each step the sum of the four terms stayed the same while the product increased or stayed the same, we have

$$\begin{aligned}abcd &\leq (T)(T)(T)(T) \\ abcd &\leq T^4 \\ \sqrt[4]{abcd} &\leq T \\ \sqrt[4]{abcd} &\leq \frac{a+b+c+d}{4}.\end{aligned}$$

So we have the inequality in four variables. Now for the proof in three variables. Suppose that a , b , and c are positive integers. We want to show that

$$\sqrt[3]{abc} \leq \frac{a + b + c}{3}.$$

Apply the inequality for four variables to a , b , c , and $\sqrt[3]{abc}$:

$$\sqrt[4]{abc\sqrt[3]{abc}} \leq \frac{a + b + c + \sqrt[3]{abc}}{4}.$$

The left-hand side above is actually equal to $\sqrt[3]{abc}$ (this is easiest to see if you convert everything to fractional exponents and simplify). Thus,

$$4\sqrt[3]{abc} \leq a + b + c + \sqrt[3]{abc}$$

$$3\sqrt[3]{abc} \leq a + b + c.$$

We've shown that

$$\sqrt[3]{abc} \leq \frac{a + b + c}{3}.$$

Try using the argument to establish an AGM for eight numbers, and then use it to derive an AGM for seven numbers.

**INTRODUCTION TO THE
MAGICAL MIX****OVERVIEW**

This section of the module focuses on an extended investigation of one problem and its variations. The problem is stated at the beginning of Investigation 6.7 in the Student Module. The introductory problem really was encountered on a multiple-choice exam by one of our students named Rich. He solved it as described, and brought it back to class to talk about. Students in your class might have examples of their own of how they outsmarted a test.

This problem is not itself an optimization problem, but extensions to it—changing the triangle from equilateral and moving the point outside the triangle, for example—generate more optimization problems. The way Rich solved the problem—using a mixture of deduction, reasoning by continuity, and looking for invariants, along with some test-taking skill—combines very useful habits of mind for solving optimization problems.

This is a reading activity to set the tone for the extended investigation that is the focus of this section of the module. You may want to ask students to read it the night before beginning this section or you can have them read it aloud in class. As a follow-up, you may want to discuss just what is meant by “mathematical habits of mind” and why they are so important.

A STUDENT OUTSMARTS THE TEST

Materials: If you plan to build a mechanical model, you will need:

- a hanger
- some string
- a washer.

See the *Solution Resource* (Problem 5 notes) for details on the mechanical model.

Technology: Geometry software is suggested for Problem 4, but you may have students experiment with a mechanical model instead. See “Without Technology” below.

OVERVIEW

This investigation introduces Rich’s problem, describes his method of “reasoning by continuity” to solve the problem, and develops two proofs of the result that the sum of the distances from any point inside an equilateral triangle to the sides of the triangle is equal to the length of the triangle’s altitude. Along the way, students will learn about functions (and constant functions), similar triangles, equilateral triangles, and inequalities.

Students need to use the Pythagorean Theorem to calculate the height of an equilateral triangle.

TEACHING THE INVESTIGATION

You might want to assign Problem 1 for homework the night before starting the investigation.

This investigation contains several sections of reading material with problems interspersed. If your class is used to reading mathematics, the various readings can be done for homework, and the problems can be solved and discussed in class. If your students have difficulty with the reading, you may want to read and discuss in class and use the problems as homework.

The investigation can be broken down as follows:

- Introduction to Rich’s problem and Problem 1
- Explanation of Rich’s solution and Problems 2 and 3
- Using software and/or a mechanical model to investigate the problem (Problems 4 and 5)
- Two proofs of the result, one algebraic and one more visual (Problems 6 and 7)
- Statement of the theorem and follow-up questions (Problem 8 and “Checkpoint” problems)

For Problem 6, it’s important that students see the potential power in mathematics of doing something like calculating area in more than one way. In this particular case, they can use the technique to prove that the sum of the distances to the sides of the triangle from *any* spot inside the triangle is equal to the height of the triangle. They may use such a technique, again, for any number of other purposes.

Problem 14 is difficult. See the module *The Cutting Edge* for several “proofs without words” of the Pythagorean Theorem, including this one.

ASSESSMENT AND HOMEWORK IDEAS

- Problem 7 assesses understanding of what students are trying to prove (just what is the conjecture anyway?) and the students’ ability to interpret pictures and “put the words on them.”
- Problem 9 is a basic assessment question that asks students to resolve the same problem with different numbers.
- Problem 13 asks students to prove a special case of Rich’s conjecture, and should involve just adapting the proof presented in Problem 7. This would be a good assessment of their understanding of that proof.

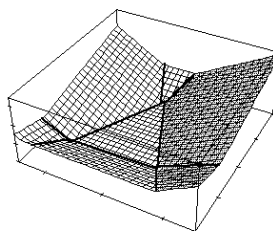
WITHOUT TECHNOLOGY

Problem 5 suggests building a mechanical device to model Rich’s problem. One such device is shown in the Solution Resource. If your class does not have access to geometry software, this is a way they can experiment with and come to understand the situation.

MATHEMATICS CONNECTIONS

Rich’s function is another example of a function defined on the plane (in this investigation, we only look at it on the interior of a triangle): It calculates the sum of the distances to the *sides* of the triangle. The fact that the function is constant on the triangle’s interior is quite surprising, but, can be proved by calculating the area of the triangle in more than one way. In Investigation 6.8, we move the point outside the triangle and see that the function is no longer constant. Can you think of an “ordinary” function (that is, $y = f(x)$, “number-in number-out”) that is constant for awhile and then starts growing? In the language of calculus, such functions can’t be differentiable

at the places where things start growing. In our case, this happens on the sides of the triangle. The surface plot of the function looks like this:



So there are distinct folds where there are no well-defined tangent planes.

VARIATIONS ON A PROBLEM

Materials: If you plan to build a mechanical model, you will need:

- a hanger
- some string
- a washer.

See “Without Technology” below for more about the mechanical model.

Technology: Geometry software is used in several problems.

OVERVIEW

This investigation begins by asking students to come up with variations on Rich’s problem by changing one or more of the elements of the problem.

Students next explore one of many possible variations on Rich’s problem—what if the point from which you measure the distances moves outside the triangle? This involves more work with functions and contour lines.

Finally, students explore a second variation of Rich’s function: Suppose we change the *triangle* on which we measure the sum of the distances to the sides. Specifically, what if we use a scalene triangle? Then the function that calculates the sum of the distances to the sides isn’t constant anymore, but that makes it all the more interesting: Where does the function attain its smallest value? Its largest value? What *are* its smallest and largest values? The important point here is that these questions can be answered by analyzing what goes wrong with any of the proofs of the equilateral result when you remove the equilateral condition. This interplay between proof and experiment is one of the driving forces behind mathematical invention.

In the course of this investigation, students will learn more about altitudes in triangles, area of triangles, and proof.

Familiarity with geometry software is necessary for this investigation. Students should be able to construct equilateral triangles, make a perpendicular from a point to a line, drag points, and measure distances. Students who have worked through Investigation 1.13 in the module *Habits of Mind* or equivalent activities will be adequately prepared.

Knowledge of contour lines and contour plots gained in Investigation 6.4 is also assumed.

To complete the proof of the generalization of Rich’s conjecture, students will need to know that in a triangle, the largest altitude goes from the smallest angle to the smallest side. This fact could be developed when needed.

TEACHING THE INVESTIGATION

A class discussion around Problems 1 and 2 would be a quick way to transition from the previous problem (Rich's problem) to the new problems presented in this investigation.

The first half of the section "Move Outside the Triangle" should be done in class because of the reliance on geometry software.

Problem 10 is difficult, and it would be an ideal challenge for a class that's working hard on ideas about proof.

Problem 15 is the key to the section "What If the Triangle Isn't Equilateral?" and should get lots of time. Problems 12–14, with some investigation on 14, could be used to set the stage for the investigation. Problem 15 should take at least one night of homework and one full class of presentations and discussion. If students work on it in groups (each group taking on only one of the proofs), they may need two class periods: one for solving the problem and one for preparing a presentation. Problems 16–18 are ideal follow-ups to these presentations; if students understand *where* the proofs go wrong, they should be able to apply that to questions about maxima and minima.

The central mathematical message in this investigation is that seeing what goes wrong in a proof when the hypotheses are relaxed gives you some insight into how the statement of the result needs to be modified. But there is another mathematical connection here: Students see that, if the triangle isn't equilateral, Rich's function always assumes values between the shortest and longest heights of the triangle (as long as you stay inside the triangle). What if these two heights are the same? Then, on one hand, the triangle is equilateral; on the other hand, the function is constant on the triangle's interior, recovering the student's original result. So often in mathematics, a result is noticed in a special case, and an explanation is constructed that makes use of the features of the special case. This leads to a consideration of more general cases, and experimentation shows how the result must be modified to accommodate more generality; new conjectures arise. Then the actual *methods* used to establish the special result can be studied, and, with a little perseverance, one can modify the arguments to establish the more general results. These general theorems, in turn, *imply* the original result that motivated the whole investigation, providing deeper insights into it and placing it in a more general context.

ASSESSMENT AND HOMEWORK IDEAS

- Problems 7–9 could be done for homework after the earlier problems are done in class.
- Problem 11 is difficult but would make a good final assessment, especially if students could work together on it.
- Problems 1, 2, and 3 of Investigation 6.10 are appropriate to do as homework or assessments after this investigation.
- Problem 6 of Investigation 6.10 is appropriate now, as well, but would be better done in class.
- Problems 12 and 13 can be done for homework, followed by investigation of Problem 14 in class the following day.
- Problem 15 is an ideal group project for presentations. Individuals should read over both proofs as homework, work in class for one or two days to find where the proofs go wrong and to prepare their presentations, and make posters and do a write-up for homework the night before their presentations.
- Problems 16–18 are good individual assessments of students' understanding of Problem 15. These could be done for homework or in class.
- Problem 19 is a special case (it limits the placement of the point to one side of the triangle). This one could be done as a quiz after Problems 15–18 or as a homework assignment. A complete write-up of this problem, including how it relates to the work students have been doing, could go in a portfolio.
- Problem 20 would make a good project. If students are having difficulty making progress, they could start by just looking at the case where $m\angle C = 90^\circ$.

WITHOUT TECHNOLOGY

The mechanical model described in the Solution Resource notes for Problem 5 of Investigation 6.2 could be adapted to explore the nonequilateral case.

**MAKING CONNECTIONS
IN MATHEMATICS****OVERVIEW**

This is primarily a reading investigation, with three problems at the end that relate both to the reading and to the mathematics of the second and third sections of this module. It is a story about Fan Chung, a mathematician who has worked on similar problems.

TEACHING THE INVESTIGATION

You may ask students to read Fan Chung's story and then have a short discussion about "what does a mathematician *do*?" This discussion does not need to be limited to careers as research mathematicians, but more generally, "what do people who study mathematics do?" The idea that people solve problems for a living is likely new and surprising.

If you plan to do the third section of the module, "The Airport Problem," with your students, you may want to have a quick exploration of Problem 1 so they connect this with their own work later. If you plan to skip "The Airport Problem," you may want to spend a little more time on the Bell Labs problem.

**USING THE MAGICAL
MIX****OVERVIEW**

This is a set of problems designed to be used in conjunction with Investigations 6.7 and 6.8. They can be used for additional problems as you teach those investigations, for homework, or for a final assessment. Some of the problems here are particularly challenging and would be good for advanced classes, for projects or presentations, or for individual students who need an extra challenge.

Students should be working through, or should have completed, Investigations 6.7 and 6.8.

ASSESSMENT AND HOMEWORK IDEAS

- The *Teaching Notes* for Investigations 6.7 and 6.8 include notes about which problems are appropriate to use with those investigations.

GETTING STARTED

OVERVIEW

This section of the module (Investigations 6.11–6.18) focuses on an extended investigation of one problem and its variations. The problem has a long and interesting history; some readings about it are included here and in the student materials.

PROBLEM

Three neighboring cities, all about the same size, decide to share the cost of building a new airport. They hire *your group* as consultants to find possible locations for the airport.

As stated, this problem is not yet an optimization problem, but the track the investigation takes—trying to find the spot that requires the minimal new roadwork (with the admittedly-problematic assumption that no existing roads are being considered)—turns it into one. In teaching this investigation, we have found that some students really enjoy the purely-mathematical exploration of minimizing distance in this pretend world. Others (perhaps future engineers?) are annoyed by the detachment from reality. They prefer to work with a real map and take existing roads, lakes, and other potential obstacles into account. This may be a much more difficult problem. The projects suggested in Investigation 6.17 are the ideal place to let these students run with their ideas.

Materials: maps

The day before: Bring in photocopies of maps—you may want to choose several different states or countries so that each group can work with a unique situation.

In this introduction, students are presented with the airport problem. They locate three cities on a map that they will use for a final presentation and make conjectures, based just on the problem and their map, about where the airport should go.

TEACHING THE INVESTIGATION

This is a good first-day activity. If you want students to work in groups, give each group a map, ask them to mark their cities, and give them time as a group to discuss where the airport would go. You can end with a whole-class discussion about the decisions made and what features they took into consideration.

If you want to downplay the “consulting group” aspect of the investigation, you could ask students to do this investigation as homework the night before beginning discussion of the airport problem in class, or you could skip this investigation altogether and just present the airport problem at the opening of Investigation 6.12.

WHAT IS MEANT BY BEST?

OVERVIEW

In this investigation, students explore the possible “best spots” for the airport. One of the possibilities, finding a place that minimizes the total length of roads that must be built, is chosen as the theme for Investigation 6.13. This goal can be stated in mathematical language: Given three points A , B , and C , for what point D is the sum $DA + DB + DC$ as small as possible?

In the process of beginning the investigation, students will learn about continuous variation, specifically the system of the triangle ABC with its internal, continuously varying point D .

Problem 2 asks students to explain how to find the airport location assuming that the airport is to be equidistant from the three cities. This is the circumcenter of the triangle. Students need not know about circumcenters ahead of time, but a small diversion into what they are and how to find them may be appropriate either before or after students work on that problem.

TEACHING THE INVESTIGATION

If you want to begin the investigation with the class discussion, Problems 1–3 could be done in class (preferably in groups) with write-ups and/or Problem 4 assigned for homework. Alternatively, Problems 1–3 could be done for homework the night before beginning the investigation, with the class discussion and work on Problem 4 taking up the first day of class.

The framing of the airport problem in the *Student Module* and the first paragraph of text following that problem are important; these should be read aloud in class.

Problems 7 and 8 are designed to lead into the next investigation. You can just touch on them briefly, explaining that the goal of the next few investigations will be to answer them. Or you could ask students to write up answers to these questions based on what they know, with the understanding that these answers will be revised later.

The important idea in this investigation is that D can be thought of as a *moving* point, and, as it roams around the plane, it carries with it a process (calculate the sum of the distances to the vertices of the triangle) that produces a *value* (the actual number you get when you sum the distances). This process view of mathematical functions is something that many high school students never develop. Using the language of

The economy solution will be the focus of this investigation and the next one.

functions and processors in contexts like this (where we never actually write down an explicit formula for the process) can help students begin to see functions as machines.

For Discussion (*Student page 83*): In the discussion of the definition of “best,” several possibilities will probably emerge. In addition to the fairness one mentioned in the Student Module, make sure to include:

- *Economy*: The total cost of the roads built from the cities to the airport should be as small as possible.
- *Pollution Reduction*: There should be as little total driving as possible in order to reduce exhaust emissions and traffic.

Students should understand that, in answering Problem 7, they are not expected to come up with the “right” conjecture at this point. The idea is to come up with something reasonable to explore and refine during the next investigation.

ASSESSMENT AND HOMEWORK IDEAS

- Problems 1–3 could be done for homework the night before beginning the investigation.
- Problem 5 could be done for homework.
- The early conjectures and explanations that are written up in Problems 7 and 8, followed by conjectures and explanations from later in this section of the module, would make ideal portfolio items.

**SPECIAL CASES AND
MODELS****OVERVIEW**

This investigation presents three ideas for investigating and developing a conjecture about the airport problem:

1. Look at simpler problems and special cases;
2. Look at a mechanical model;
3. Use a computer.

Overviews of each idea follow.

In the first one, “Look at Simpler Problems and Special Cases,” students use ideas about continuous change. Special cases are investigated: if there are only two cities, the airport can go anywhere between them; if there are three collinear cities, the airport should go at the middle city; if the middle city is moved off the line between the other two by just a little, the ideal spot for the airport shouldn’t move by very much; if one city is very far from the other two, the airport should be close to the segment containing the two close cities; and, if the cities form an equilateral triangle, the airport should go at the “center.”

In “Look at a Mechanical Model,” three devices are suggested that allow students to invent and check conjectures. These models have the potential to help students think about the “sum of the distances function” as a continuously-varying system. They also help convince students that there is a unique best spot for the airport. The three devices are:

- a circle of nails in a wooden board (or a circular geoboard), with string to model the total distance between three cities and a fourth point modeled with a small metal ring;
- soap films;
- holes drilled in a board, with string passing through the holes and pulled taut with weights.

In “Use a Computer,” students use geometry software to model the airport problem. Essential to this technique is the building of a dynagraph so that students can observe the sum of the distances to the airport both numerically and geometrically. The model is also used to approximate the contour lines for the airport function.

Materials: Depending on which model(s) you decide to build, you will need:

- wooden board, nails, and a hammer (or a circular geoboard); string (or fishing line); and a small metal ring.
- clear plexiglass, a drill, nuts, and bolts; dishwashing liquid, water, and a pan.
- a wooden board, a drill, lots of string, and equal weights.

The first model gives a simple way (that is completely analogous to the pin and string construction of the ellipse) to draw the contour lines (the curves on which the sum of the distances to the cities is constant) for the airport function.

Technology: Geometry software is useful for “Look at Simpler Problems and Special Cases” and necessary for “Use a Computer.”

The night before: Do you need to reserve a computer lab? Do you want students to build mechanical models in class, or will you build them before class? Even if you decide to use computers, think about using the mechanical model, too. In this case, the mechanics adds a feel for the problem that geometry software environments can't match.

What's coming up? In the next investigation, students construct 120° gadgets to find the best spot, so they should have come up with a conjecture about 120° angles before moving on to that. Also, Investigation 6.15 presents two proofs of the solution to the airport problem.

Familiarity with geometry software is necessary for part of this investigation. If students are to model this situation with a computer, they should be able to measure and calculate with distances, drag points, and copy distances to build a dynagraph. Students who have worked through Investigation 1.13 in the module *Habits of Mind* or equivalent activities will be adequately prepared.

Two of the suggested experiments assume the knowledge of contour lines and contour plots gained in Investigation 6.4.

TEACHING THE INVESTIGATION

Depending on your class, on the materials available, and on the time you want to spend, you may want to do one, two, or all of these activities. We have found in field tests that students are often convinced of their conjecture after just one experiment and they don't see the point of performing another to come up with the same conjecture.

If you have the resources, you may want groups of students to investigate just one of the ideas, making sure that each idea is covered by at least one group. One day of class presentations should be enough for everyone to get the idea of the other investigations and to see that they all came up with similar conjectures through different methods.

Alternatively, you may want students to investigate either the mechanical model or the computer model in class and use the "Look at Simpler Problems and Special Cases" problems as homework assignments. Teaching suggestions for each activity follow:

Teaching Idea 1. The focus of these problems is to get students to see the continuity in the situation and to use it in developing a conjecture. The problems can be done either on a computer or with pencil and paper. Problems 1–3 could be done the first day (or night), Problems 4–5 the second day (or night), and Problems 6–8 are a way for students to write up their conjectures at this point and could be used as an assessment before moving onto the next investigation.

Problem 5 is a good place to bring up the transition from one case to another. With three collinear cities, the airport is best located at the center city. In an equilateral

triangle, the airport goes in the center. If you start with three collinear cities, with one city at the midpoint of the other two, and gradually move the middle city out along the perpendicular bisector of the segment between the other cities, at what point does the airport “leave” the middle city and move into the triangle’s interior?

Teaching Idea 2. This activity should be done in groups; it’s too difficult to handle the models on an individual basis. After students have built the first model, they should try several combinations of three cities, keeping track of their data in different-colored pens or on separate sheets of paper. Students could answer Problems 9–11 in class or for homework. They can then move ahead to the contour lines (this should take one additional day of class time), one of the other mechanical models (one day each of class time, plus time to build the models), or answering Problems 6–8.

Teaching Idea 3. Students should first draw the three cities, connect each to a moveable point D , and sum the three distances to D . (This sum can be seen visually with a dynagraph, but the investigation can also be done just by watching the numbers.) As they move D around, they should notice that the sum changes continuously—there are no big changes in the sum for small changes in the location of D . That’s useful for approximating the best location. Once students think they have found the best spot for D for a given configuration, they should change the triangle and start moving D again.

Students will probably need guidance to know when to change D and when to change the triangle.

To make a conjecture, students will need to keep track of the best location of D for various configurations. This can be done with printouts or by placing tracing paper on the screen and copying the final setup.

After gathering data about best locations for D , students can answer Problem 18. If they have no conjectures, they can go on to Problem 19 (which provides a big hint). Problem 17 is a way for students to become convinced that there is a unique solution, but they may have difficulty finding the points if they have not used the first mechanical model. At the end of the computer investigation, students should answer Problems 6–8.

As engaging and enlightening as these “analog gadgets” are for many people, there are two caveats that need to be made explicit:

- Many students (and adults) have a difficult time dealing with mechanical devices. They get bogged down in the details of the string and nails, and they have a difficult time manipulating the physics of the situation. For these students, descriptions of the mechanical devices are sometimes more effective (as are computer simulations

like the one in the next investigation). They can then conduct thought experiments to get at the essential messages that the physical experiments try to convey.

- Although we have much anecdotal evidence from our own classes, we are unaware of conclusive research that shows that devices like this help students construct the essential mathematics involved in the continuity properties of functions.

For Problem 6, students should just explain what work they have done so far and perhaps present the experimental evidence they have. If their method only works for certain triangles at this point, that is fine; they should just make a note of that here. In some sense, the purpose of this problem is for students to take time to organize their thoughts and their work.

For Problem 7, students might argue something like this: “If you move D closer to A , the distance to A gets shorter, but the distances to B and C each get bigger by a bit, so things won’t change by much.” That’s the appropriate level of precision for this problem. As they build mechanical and computational models in the next investigations, their feeling for the continuous nature of this problem should strengthen, and you may want to revisit this problem.

Problem 8 is difficult. In the section “Look at a Mechanical Model,” students might develop a plausible argument for their position. Over the course of this section of the module, students will *prove* the existence of a unique minimum and show how to locate it.

ASSESSMENT AND HOMEWORK IDEAS

- If you are using portfolios, Problems 7 and 8 would be a good “midpoint” entry, after Problems 7 and 8 from the last investigation.
- Any of the mechanical models could be built at home by enterprising students as a project or for extra credit.
- For Ideas 2 and 3, students can do the investigations in class, keeping notes, drawings, and printouts. The problems can be done as homework.
- Problems 6–8 should be answered as a midpoint assessment, no matter which investigation students carried out. The methods students used in Problem 6 should not rely on the mechanical models or on the computer. If they do, students will need to refine their method in the next investigation.
- Problems 7 and 8 of Investigation 6.18 could be done during this investigation.

MATHEMATICS CONNECTIONS

Problems 7 and 8 get at some important and subtle mathematical ideas. Problem 7 asks students to argue that the airport function is continuous on the plane. The precise definition in this case would demand that you can make two values of the function (that is, two different sums of distances to the vertices from two different points) as close as you like (on the number line) by making the points at which you calculate the sums close enough on the plane. So, by making two airport locations close enough, you can make the associated sums of distances as close as you like (say, making their difference less than $\frac{1}{1,000,000}$). In analysis classes in college, students spend their time calculating things like just *how* close you'd have to make the airports in order to make the sums of distances to the vertices within $\frac{1}{1,000,000}$ of each other (to complicate matters, the answer to this question probably depends on where you are on the plane relative to the cities).

Investigation
6.14

Student Pages 96–99

TESTING THE CONJECTURE

Materials: Depending on which of the 120° gadgets you want to make, you will need:

- transparencies, scissors, and pushpins,
- toothpicks and clay,
- paper clips.

Technology: Geometry software can be used to build one of the 120° gadgets.

OVERVIEW

At the beginning of this investigation in the Student Module, an explicit conjecture is stated, not in complete generality, but precise enough to deserve a theoretical investigation.

Students discuss the conjecture (and how to make it more precise) and then they develop several methods to *locate* the theoretically-best spot in any triangle. One of the ways to do this is a consequence of (and, perhaps, actually motivated) a proof that the Fermat point is usually the best spot to put the airport. This proof is taken up in the next investigation.

We want the students to have come up with their own 120° conjecture before beginning this investigation. In fact, their conjectures may be stronger than the one presented in the Student Module, but we didn't want to give away too much.

Problem 4 requires building equilateral triangles on the side of a given triangle (either with geometry software or with pencil and paper).

TEACHING THE INVESTIGATION

This is a short, 1 to 2 day investigation. Students build “ 120° gadgets” and practice finding the best airport location in several different triangles. They may need some guidance to look at obtuse triangles and to try to find the cutoff point—where does the 120° spot no longer seem best?

The “Checkpoint” problems provide a review of work done so far on the airport problem.

Locating the theoretically best place for the airport can be suggested by mechanical devices, but the construction in Problem 4 deserves special attention because it provides the basis of the proof of the conjecture in the next investigation. Work on Problem 4 might be enhanced by the various devices for locating the Fermat point.

ASSESSMENT AND HOMEWORK IDEAS.....

- Problem 5 (asking students to refine the conjecture) is a good assessment to see whether students have been following the experiments and to watch for places where things go wrong.
- Problem 9 is a good assessment of the students' understanding and use of contour lines.
- For a class working on proof, proving the conjecture in Problem 4 is a good challenge. It requires triangle congruence, angle sums in triangles, and some persistence.

Technology: Using geometry software makes the key ingredient to the proof seem more natural.

Make sure students recall their solutions to the burning tent problem, since Hoffman's proof is compared to it.

ESTABLISHING THE CONJECTURE

OVERVIEW

This investigation presents two proofs of the airport conjecture, one worked out carefully and the other left for students to work through. The first proof is used to give an algorithm for locating the Fermat point. Its construction also sheds some insight on why the Fermat point is not the best spot for the airport if one angle of the triangle is bigger than 120° . (The Fermat point falls outside the triangle in these cases.) This proof sets up a construction in which the airport's location is found by drawing a segment between two points in order to minimize the distance between them. This useful habit of mind is reminiscent of many shortest path problems from Investigation 6.2.

The following knowledge is needed for understanding the proofs: facts about equilateral triangles, angles about a point (especially vertical and straight angles), facts about congruence (rotating a triangle produces a congruent one, for example), and measures of inscribed angles. Any of these facts can be developed or postulated (for later analysis) during the course of this investigation without causing a serious detour.

TEACHING THE INVESTIGATION

This investigation involves a great deal of reading, beginning with the historical essay and continuing through the two proofs. We have found that students who have difficulty reading mathematics do better if they take turns reading aloud (either as a whole class or in a small group) clarifying questions as they go. This investigation may therefore take several days. Here's one possible lesson plan:

Day 1: Ask students to read the "Perspective on Fermat" the night before. On the first day, read the introduction to Hoffman's proof aloud as a whole class. As a whole class or in smaller groups, have students read through Hoffman's proof. End with discussion and questions. For homework, assign Problems 1–3.

Day 2: Discuss Problems 1–3 and Hoffman's proof as a whole class. In small groups, work on Problems 4, 6, and 7. At the end of class, discuss problem 5. For homework, students prepare their presentations.

Depending on the time available, you may want some groups to present Hoffman's proof and others to present Toricelli's. Those presenting Toricelli's have more of the proof to construct on their own, but they have the benefit of having worked through another proof first.

Day 3: Students make presentations of Hoffman's proof. For homework, assign Problems 8–10, 12, and 13. (11 is optional.)

Day 4: In groups, work on Problem 14. Homework is Problem 15 and preparing presentations of Toricelli's proof.

Day 5: Students make class presentations of Toricelli's proof. For homework, assign Problems 3–5 of Investigation 6.18.

Day 6: In-class assessment about the airport problem: What is “the answer” to how to find the spot that minimizes total distance from three fixed points? Explain how to find this spot. Explain how you know this method works.

There is a long-standing tradition in graduate schools of mathematics in which graduate students study famous proofs of theorems and then present the proofs to classmates. This is very beneficial. Just as studying a piece of music is an important ingredient in developing skill at composition, studying the proofs of others helps one develop skill at constructing proofs. Simply memorizing the proof presented here is useless, but reworking the proof and presenting it in your own words could be quite valuable. An especially important exercise is to give some scenarios that might have led Hoffman and Toricelli to the *invention* of these proofs in the first place.

ASSESSMENT AND HOMEWORK IDEAS

- See “Teaching the Investigation” above for specific ideas, including a final assessment (under “Day 6”).
- Any of the problems in Investigation 6.18 can be used by the end of this investigation.
- There are any number of research questions you might ask the students to work on. Here are three examples:
 1. Diophantus authored the number theory text Fermat was reading when he wrote his famously cryptic message. Find out and report upon some of the work for which Diophantus is famous.
 2. Besides Fermat, Steiner, and Hoffman, a few of the many other known mathematicians who worked on the airport problem are Gauss, Toricelli, Fesbender, and Viviani. Recorded history reports results for these men. Find out who some of them are.

3. Do you think there may have been other people, whose names do not appear in our history books, who worked on and arrived at original results for the problem? Speculate on some of the reasons why any one person's work might not have been recorded or might not have received enough recognition to receive a place in the research of that time.

USING TECHNOLOGY

You may want students to use geometry software while working through Hoffman's proof. While not essential, this is *very* effective, especially if done on an overhead projector screen.

THE AIRPORT REVISITED**OVERVIEW**

This investigation connects the airport problem to Rich's function (from the second section of this module), which gives the sum of the distances to the sides of an equilateral triangle. The idea here is for students to see connections between the two results. (If you skipped the section "A Magical Mix," you should skip this investigation, too.)

Technology: Geometry software is useful for Problems 2 and 5.

Students should have already worked through both the second and third sections of this module. The sum of the angles in a quadrilateral is a key piece of a proof in this investigation.

TEACHING THE INVESTIGATION

Students can read the first page of the investigation either the night before or the first day of the investigation. Problems 2 and 3 are not difficult, but students working in groups are more likely to come up with the key insight. Problem 4 is the key to connecting the ideas from Rich's problem and the airport problem. If students do Problem 4 for homework, you will certainly want to discuss it in class the next day. Problem 5 could be done as an in-class assessment. Reading the "Perspective on the Steiner Problem" in class is a nice way to end the investigation.

Problems 2 and 3 may appear harder to students than they actually are once students are involved in them. The key insight is that the angle sum in any quadrilateral is 360° , and that 180° of that is taken up by two sets of perpendicular segments.

ASSESSMENT AND HOMEWORK IDEAS.....

- If students have worked through all of "The Airport Problem" up to this point, any of the problems in Investigation 6.18 is appropriate, particularly Problem 1.
- Problem 4 can be done for homework.
- Problems 5 and 6 make good in-class assessments.

PROJECTS RELATED TO
THE AIRPORT PROBLEM

Materials: Whatever materials students used in their previous investigations should be available to them now.

Technology: If students have been using geometry software, they may want to continue using it here.

OVERVIEW

This investigation includes three project ideas. Students (individually or in groups) can select one of the ideas and explore it on their own, using some of the techniques they've learned while investigating the airport problem. These are not the only possible projects, and students should be encouraged to come up with their own question to investigate.

1. Students prepare a presentation to the city councils of their three cities, explaining how to locate the mathematically-best spot for the airport, giving a proof of this fact, demonstrating computer experiments, and then explaining whether they chose to put the airport at the theoretically-determined optimal point and why.
2. Students explore the following question: What is the best place for an airport serving four cities? The first thing that's necessary is that students reassess their concept of "best."
3. Students investigate related problems, like networks and soap bubbles.

Students should have worked through Investigations 6.11–6.16 before beginning this one.

TEACHING THE INVESTIGATION

Teaching a "project" unit presents special challenges—providing the right amount of direction without giving too much, helping students decide on a final product, and keeping a class together and moving forward when students aren't all working on the same thing. Here are some tips:

- Set deadlines for rough drafts. If students have one week to complete the project, they should give you an outline that includes their idea for a final product within two or three days.
- Have all materials out and available. That way students can help themselves, and you don't need to spend time finding materials in the middle of class.
- Check in with each group every day, even for just a minute, to make sure they're making progress.

ASSESSMENT AND HOMEWORK IDEAS

- This entire investigation is an assessment. A final, polished product (presentation, paper, etc.) should be the goal. This may require two or three drafts with feedback from you.

WITHOUT TECHNOLOGY

Another project idea would be for students to build and explain any of the mechanical models from Investigation 6.13 that the class hasn't already used.

USING IDEAS FROM THE AIRPORT PROBLEM

OVERVIEW

This set of problems has been designed to be used in conjunction with Investigations 6.12–6.16. They can be used for additional problems as you teach these investigations, for homework, or for a final assessment.

Students should be working through, or have completed, Investigations 6.12–6.16.

ASSESSMENT AND HOMEWORK IDEAS.....

- Each of Investigations 6.12–6.16 includes notes in the “Assessment and Homework Ideas” section as to which problems are appropriate to use with that investigation.

**THE ISOPERIMETRIC
PROBLEM****OVERVIEW**

This section of the module focuses on one of the most famous optimization problems:

PROBLEM *The Isoperimetric Problem*

Of all closed curves with the same perimeter, which one has the most area?

Investigation 6.19 helps students to construct a proof that the solution to this problem is a circle. Investigation 6.20 presents problems based on the long and interesting history of this problem. Investigation 6.21 presents some modern mathematics in related problems.

Depending on the time you want to spend on this section, you might decide to do only Investigation 6.19, or to do Investigation 6.19 and either Investigation 6.20 or 6.21. Doing all three investigations could take between two and three weeks of class time.

The goal of this investigation is for students, with some guidance, to create a proof of the isoperimetric conjecture. This involves taking a three-step outline of the proof and proving the last two steps. (The first step will be a hypothesis of the proof.)

Students need to be able to calculate and compare areas in various ways. These include counting grid squares, fitting one shape inside another, and using area formulas. Students will also reason about the effect or lack of effect on the comparability of the areas of two shapes given certain properties, such as the variability of perimeter, or convexity, concavity, and equal perimeter.

TEACHING THE INVESTIGATION

This investigation involves a lot of reading and has an emphasis on proof. Depending on your students, this may mean that much of the investigation will be covered in class and perhaps less will be assigned for homework. Here is one possible plan for a class that needs moderate guidance in the reading:

Materials: Rope or string make ideal manipulatives for exploring Problems 2 and 14.

Day 1: Read the first page and work in groups on Problems 2 and 3. For homework, do Problem 4 and read “The Plan.”

Day 2: Discuss the role of proof (Problem 5) and “The Plan.” Read the section following the “For Discussion” on page 126 aloud (either in small groups or as a whole class). Begin work on Problems 6–8. For homework, finish and write up Problems 6–8.

Day 3: Read and discuss the “For Discussion” on page 128. Read through “Ways to Think About It,” and begin work on Problems 10–14. For homework, do Problem 9 and finish (and write up) Problems 10–14.

Day 4: Discuss “Step 1” (see “Notes” below for important ideas in this discussion). Work on Problems 15 and 16. For homework, write up Problems 15 and 16. Read “Step 2.”

Day 5: Work on Problems 17–20. Discuss the “For Discussion” on page 132. For homework, write up Problems 17–20. Also do Problem 23.

Day 6: Read “Step 3” in small groups, and do Problems 22 and 23. For homework, start Problem 24 (see “Assessment and Homework Ideas” below).

One way students might approach Problem 5 is to think about how this emphasis on proof in mathematics is different from their other classes. Do they prove their conjectures in an English paper, or does it suffice to provide facts which support their opinions? Is it OK to make conjectures in a math class? Questions such as these may help students to think about the role of conjectures, theorems, and proofs in mathematics. They may not have even realized, in fact, that there are always new mathematical proofs being produced. It is not uncommon for students to see mathematics as a finished subject, where there’s nothing left to discover or prove. This is far from true!

A theorem from calculus is relevant to Problem 6: A continuous function on a closed and bounded interval must obtain a maximum and minimum value. If the interval is open, however, the function may or may not have a maximum or minimum.

The idea in Problem 8 is for students to look beyond the algebraic details and get the idea behind the proof: We assumed that x was the largest number smaller than 1, then used this to perform some calculations, and discovered that x must then be bigger than or equal to 1. This doesn’t make sense, so it must be that no such x really exists; thus, there is no largest number less than 1.

The main technique throughout the proof will be to suppose this best possible curve is some curve other than a circle, and then to find another curve with the same perimeter but more area, contradicting that the original curve was the best one.

For Discussion (*Student page 128*): It's important that students understand the way we are treating Step 1. We don't want them to have the mistaken assumption that Step 1 is an unsolved problem; we just want them to understand that the proof of this step is most likely too difficult for them at this time. Here's an opportunity to point out to them that this is often how mathematicians work; if a certain step is too hard to prove right away, mathematicians will assume the result, proceed with their work to see where it gets them, and come back to prove that step if it is indeed needed.

To see how Step 1 is used, look back at the way Steps 2 and 3 are worded; they both say “given a curve that encloses the most area.” The statements assume a best curve exists; to prove these steps, we take advantage of the fact that the curve under consideration is the best possible curve.

Any plan for a proof which doesn't require showing the existence of a best curve would have to be set up differently. In the proof we present, we start with the knowledge that the curve exists, and then we discover more and more information about the curve until we can conclude that it's a circle. A different method for a proof would start out with a circle of a given perimeter and then make use of the properties of a circle to prove it contains more area than any other curve with the same perimeter. As is mentioned later in the Student Module, our proof requires more of a detective method; we want to deduce that the best curve is a circle.

For Problem 10, the following theorem explains what goes wrong:

THEOREM

A geometric series converges if and only if the absolute value of the ratio is less than 1.

In this example, the series diverges, so x is not a finite number which you can work with. Thus, all the remaining calculations are invalid. The class does not need to be introduced to the notion of a geometric series, but see “Mathematics Connections” below if you are interested in taking a detour into this territory.

Problem 13: Refer back to the Student Module for problems related to the following theorem: “Of all the polygons with a given perimeter and a given number of sides, the regular polygon has the most area.” Students are asked to apply a theorem from the Student Module for this problem—but they can discover the answer themselves.

Ask them to play around with several quadrilaterals to try to understand this theorem for themselves. An example is shown below:

Recall that you already know that, given a fixed perimeter, the rectangle with that perimeter which encloses the most area is a square.

Consider a parallelogram of sidelengths a and b (remember that opposite sides of a parallelogram are congruent), and compare it to a rectangle of sidelengths a and b . Both polygons have perimeter $2a + 2b$. The area of the rectangle is ab .

Given any parallelogram, the rectangle with the same dimensions encloses more area.

Consider the parallelogram. Call the height h . We know that $h < b$ because h is the length of a side of a right triangle, while b is the length of the hypotenuse of that triangle. So the area of the parallelogram is

$$\text{Area} = ah < ab.$$

But this says that the area of the parallelogram is always less than the area of the rectangle. This means that, given any parallelogram, the rectangle with the same perimeter encloses more area, so the best curve can't be a parallelogram which is not a rectangle. We also know that the square with the same perimeter as the rectangle encloses more area, so an oblong rectangle can't be the best curve either. The conclusion is that "if the best curve is any type of parallelogram then it must be a square."

We only looked at parallelograms above, but students should do similar experiments on their own. For example, you can also employ a piece-of-string argument to show that, given an arbitrary quadrilateral, a rhombus with the same perimeter is better. Take a quadrilateral, fix two sides and think of the sum of the lengths of the other two sides as a piece of string pinned down at two vertices. Where can you put the remaining vertex to maximize area? It turns out that the area will be the largest if you place the vertex so that you form two congruent sides. Continue this way, looking at two sides at a time and showing they have to be congruent. Hence, you will show the largest area is enclosed by an equilateral quadrilateral, which is a rhombus.

Of course, these can all be done in general for radius r or sidelength s , but students may feel more comfortable working through examples with numbers first. A single example is enough to show that squares are not, in general, the best shape.

If it didn't, there would be a curve of the same length with endpoints on the diameter but enclosing more area. Well, then we could flip this curve about the diameter to obtain a closed curve with more area than our best closed curve, which isn't possible.

For Problem 14, students can show that a square is less optimal by computationally comparing squares to circles (which they conjecture are better). Pick the radius of the circle $r = 1$. The area is π , and the circumference is 2π . A square with perimeter 2π would have sidelength $\frac{\pi}{2}$ and area $\frac{\pi^2}{4}$, and $\frac{\pi^2}{4} < \pi$. Alternatively, pick a square of sidelength 1. Then the area is 1, and the perimeter is 4. Similar computations will show that a circle with circumference 4 will have area $\frac{4}{\pi} > 1$. Note these examples do not show that a circle is best, just that it is better than a square.

For Discussion (*Student page 132*): Suppose we have a closed curve which is the solution to the isoperimetric problem. Then the top portion of the curve must solve the reduced isoperimetric problem. Likewise, suppose we have a curve with endpoints on a given line which is a solution to the reduced isoperimetric problem. Then the closed curve that's formed by flipping this curve about the line is a solution to the isoperimetric problem. This means that, given a solution to one of the conjectures, we can obtain a solution to the other. The utility of the reduced isoperimetric conjecture is the fact that it allows us to look only at curves with their endpoints on a given line, as opposed to working with closed curves.

ASSESSMENT AND HOMEWORK IDEAS

Problem 24 is the obvious final assessment for this investigation. You may want to allow students to write more than one draft of this proof, with feedback from you between drafts, so that they can present a polished final product. A list of elements students should have in their proofs is included in the Solution Resource. Students' proofs should contain complete sentences, pictures, and formulas as needed. Here is an outline for such a proof:

- State the problem.
- State the existence hypotheses.
- Eliminate curves which are concave or which are not simple. (Here students can use the arguments and pictures from Problems 11 and 12.)
- Show that the best curve has a diameter which bisects the area. (Students can use Problem 17. The argument on page 131 in the Student Module shows why this diameter must also bisect the area.)
- Explain the reduced isoperimetric conjecture.
- Explain the right angle argument. (Basically students need to repeat the work and pictures from Problem 23.)

- Show that the best curve is a semicircle. (Students must show why Step 3 implies that a semicircle is the solution to the reduced isoperimetric problem.)
- Conclude the proof. (This is where students explain that the solution to the reduced isoperimetric problem is a semicircle, and why this implies the answer to the isoperimetric problem is a circle.)

See “Teaching the Investigation” for a possible homework plan. The idea is that students will do reading and work on problems in class. For homework, they will write up problems, perhaps do one or two problems on their own, and do small amounts of reading. If your class is comfortable reading mathematics, you may want to change things around so that more of the reading happens at home.

MATHEMATICS CONNECTIONS

In Problem 11, students should be willing to accept the idea that “an infinite number of numbers added together doesn’t necessarily yield an infinite number.”

It’s unfortunate that mathematics has adopted the phrase “infinite sum”; students are quite right to be suspicious of a process that requires an infinite number of steps. What we *mean* when we say “an infinite sum is such and such a number” is that the *partial sums* we can obtain can be made as close as we want to that number by adding on enough terms. So, saying that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

is the same thing as saying that you can make the sum of

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n}$$

as close to 1 as you want by making n large enough. All the talk about “infinitely many terms” and “infinite sums” is poetic shorthand for looking at sequences of finite sums. (Mathematics can be fond of using fancy language for simple things.)

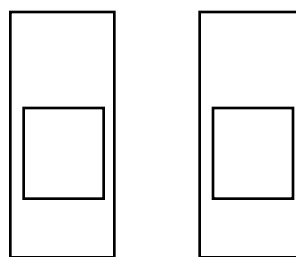
If students are familiar with geometric series, an interesting exercise is to reflect on why, in some cases, the partial sums approach a finite number, no matter how many terms are added on. After all, this seems to be a very counterintuitive idea. Below is a quick classroom activity that may provide some insight on the idea:

Ask each student to start with a regular sheet of paper. You can model the process in front of the class, so have a piece yourself. In the following few paragraphs, the

directions are written as you might say them to your class:

Tear your sheet into three pieces, to represent thirds. (There's no need to be really precise about the relative sizes.) Place two of the pieces in front of you a few inches away from each other, so you can begin to make two separate piles of paper. Take the third piece and tear it into thirds (each of which is now $\frac{1}{9}$ of the original sheet). Again, place one piece each on top of each of the $\frac{1}{3}$ pieces in front of you, and keep one piece in your hand.

After two iterations, there are two piles of two pieces each.



Now tear this remaining $\frac{1}{9}$ -size piece into thirds. As before, place one piece on top of each of the piles you've started, and keep the last piece to tear into thirds again. Keep going until the pieces are too small to work with.

Describe what's happening to the piece in your hand—what size or fractional part is it approaching? And what's happening to the two piles in front of you? (Look at just one of the piles; describe the fractional parts each piece in the pile represents, and write an expression for the sum of those pieces.)

Provided, of course, that we know what we mean by

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$$

Again, it's just the limit of the sequence of partial sums.

This is a very geometric (and very convincing) argument that

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \frac{1}{2}.$$

Similar paper-ripping arguments can show that

$$\begin{aligned} \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots &= 1, \\ \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625} + \cdots &= \frac{1}{4}, \\ \frac{2}{5} + \frac{2}{25} + \frac{2}{125} + \frac{2}{625} + \cdots &= \frac{1}{2}, \end{aligned}$$

and so on.

This could turn into a multiday investigation if you are willing to spend some time on geometric series. Your students could work on coming up with a formula for the sum, based on the size of the pieces you tear and how many of that size go into each pile. This can also be used to explain the difficulty with Problem 11. Ask students to explain what the paper ripping tells them about what happens with geometric series when the ratio between terms is greater than or equal to 1.

A PROBLEM WITH A LONG HISTORY

OVERVIEW

This investigation presents short essays on the history of the isoperimetric problem, which was solved by students in Investigation 6.19. Each essay is followed by related problems.

Students should have worked through all of Investigation 6.19 about the isoperimetric problem.

TEACHING THE INVESTIGATION

This investigation, like the previous one, requires a fair bit of reading, though in this case it comes in smaller “chunks.” Here is one possible teaching plan:

Day 1: Read Dido’s story and do Problem 1; read Aristotle’s story and do Problem 2. For homework, write up Problems 1 and 2; read Archimedes’s and Ptolemy’s stories and do Problem 3.

Day 2: Discuss Problems 1–3. Read Zenodorus’s story and do Problems 4–6. For homework, do the last reading and Problems 8 and 9.

Day 3: Write an answer to Problem 7 in class.

For Problem 3, students should find that given any polygon, you can always find another one that has the same perimeter, more sides, and larger area. See the *Solution Resource* notes for Problem 16 in Investigation 6.19.

ASSESSMENT AND HOMEWORK IDEAS

- See “Teaching the Investigation” above for one homework plan.
- Any of these problems can be used for homework. You might base your choice on the amount of reading required and how your class responds to reading mathematics.
- Problems 8 and 9 can be done for homework at any time during the investigation; they are not tied to the essay.

MATHEMATICS CONNECTIONS

A Little More History In the 6th century, Simplicios wrote a commentary on Aristotle's *De Caelo*. Referring to writings that were already lost even at that date, he reports that Archimedes (287–212 B.C.) and Zenodorus (around 200 B.C.) proved that “of [three-dimensional] isoperimetric figures, the more spacious one... [is] the sphere.” In fact, Archimedes and Zenodorus considered only a small class of solids, including, of course, the Platonic solids. See *The Ancient Tradition of Geometric Problems* by Wilbur Richard Knorr (Boston: Birkhauser, 1985).

In May of 1697, Jakob Bernoulli (1654–1705) published an especially complicated isoperimetric problem, apparently with the deliberate aim of embarrassing his brother, Johann (1667–1748). The Bernoullis were a family that produced many great mathematicians, most of whom (some say) fought bitterly with each other on both personal and mathematical grounds. So it is said that Jakob Bernoulli published his problem as a challenge to Johann, who proceeded to solve it, but his solution, unlike Jakob's, turned out to be flawed. In 1718, however, Johann Bernoulli was finally vindicated, when he developed and published an improvement to Jakob's original solution. The problem is quite complicated, but it's a generalization of the isoperimetric problem we considered in Investigation 6.19. The Bernoullis eventually used the recent development of calculus to analyze a great many optimization problems.

Another mathematician, who attempted several times to persuade Steiner that his proofs for the isoperimetric theorem were incomplete without a proof that a largest curve exists, is the German-born Dirichlet (1805–1859). He was interested in many other questions as well. He showed that sequences like 3, 7, 11, 15, ... (arithmetic sequences whose members don't all share a common factor) contain infinitely many prime numbers. To do this, he had to develop some mathematical ideas that are important research tools even today.

Despite Dirichlet's attempts to persuade Steiner that his proofs were incomplete, Steiner continued to claim that the existence of a curve containing maximum area was “self-evident.” Finally, in 1842, Steiner admitted that this unstated assumption should at least be explicitly stated, adding that the proof is thereby “readily made.” Over the years from 1870 to 1909, the hole in Steiner's 1838 proof was gradually filled by a variety of mathematicians who proved that a curve that contains maximum area must indeed exist. The first complete proof was due to Hurwitz (1859–1919) in 1901.

A whole book has been written about their arguments; the English translation of its title is *The Polemic Writings of Jacob and Johann Bernoulli on the Calculus of Variations*.

For Problem 3, we include below an algorithm which was written by several students at the University of Kansas in Professor Judy Roitman's class. The students were Stephanie Childs, Hannah Fitzsimmons, and Vincent LaVergne. This algorithm shows how to start with an arbitrary polygon and construct a series of new polygons which have more sides and greater area, but the same perimeter as the original. This demonstrates that a polygon cannot be the best curve, as new polygons are always found which enclose more area. This method is also nice because, after a few steps, it becomes clear that these polygons with greater area are starting to look more and more like circles. It is easy to play with this algorithm using geometry software; this would be a fun activity for students. Here's the algorithm:

Start with any polygon with a small number of sides. Find a line segment l which bisects the perimeter of the polygon. This segment will divide the polygon into two pieces. Now we have two cases:

Case 1: If one of the two pieces has larger area than the other, reflect that portion of the polygon about l . This will yield a new polygon with the same perimeter as the original, but with more area and possibly more sides. Now go back to the beginning and repeat the process on this new figure.

Case 2: If line segment l bisects the area as well as the perimeter, see if reflecting either one of the two pieces of the polygon about l will result in a new polygon which has a concavity. If it does, construct the new polygon resulting from this reflection. It will have the same perimeter as the original and possibly more sides. We know that we can then remove the concavity to obtain yet another polygon with the same perimeter and more area.

If neither of the two cases applies, that is, if line segment l bisects the area and you can't reflect either half about l to get a concave polygon, then choose a different line segment which bisects the perimeter and try again. Even if every line which bisects the perimeter also bisects the area, it should always be possible to find a line bisecting the perimeter with the property that reflection of one of the two pieces about the line will yield a new polygon with a concavity. All that is needed is that the line segment must intersect a side of the polygon at some angle greater than 90° .

THE RESEARCH CONTINUES TODAY

Materials: If you don't use geometry software, you will need some materials for Problems 2 and 8. Problem 2 can be done with compass and ruler constructions. Rods of fixed length, preferably with hinges, would be ideal for Problem 8.

Technology: Geometry software is recommended for Problems 2, 7 and 8.

OVERVIEW

This final investigation presents problems designed so that students can see the type of work mathematicians do, and how this work, though very difficult, is related to the problems that they have been thinking about. Don't be confused by the purpose of the research problems mentioned, such as the two-area, three-area, and two-volume problems. They are not meant to be solved by the students. The numbered problems are for them to do, as always, but that is all.

Students should have worked through all of Investigation 6.19. The content covered in the student problems includes angles between arcs and cyclic polygons. Students need not know about these things before beginning the investigation.

TEACHING THE INVESTIGATION

This investigation, like the previous two, is relatively heavy on reading. Here is one possible teaching plan, which allows students to do some of the reading in class and some on their own:

Day 1: Read the introduction and discuss Problem 1 as a whole class. Individually or in small groups, work on Problem 2. For homework, read up to the "Perspective on Student Research Experience."

Day 2: Discuss Problem 2 and the reading. Read through the "Perspective on Student Research Experience" and answer Problems 3–5. For homework, work on Problems 6–8.

Day 3: Complete Problems 6–8.

ASSESSMENT AND HOMEWORK IDEAS

- If students know the answer to Problem 1, then Problem 2 can be done for homework using compass and ruler instead of geometry software.
- Problems 3–5 are good homework problems.

- As a final assessment for this section of the module, have the students reread Investigation 6.19, restate one of the problems, and write up a detailed solution. This would be an ideal item for a portfolio. (You might use the discussion question that follows the outline of “The Plan,” Problem 9, the discussion question immediately after that, or Problems 18 or 26.)

INTRODUCTION TO OPTIMIZATION

A spreadsheet, Logo, or other software environment that allows you to make tables (graphing calculators, for example) would be helpful in this problem.

A Logo program could be used to compute the answers. It's the process of *creating* the procedures in the program, rather than just using them, that helps one understand the calculations that are being made. Three programming alternatives are suggested in the *Teaching Notes*.

Problem 1 (Student page 2) By trial and error, it looks like the best way to pack the greatest amount of peanut butter (given the two jar sizes) is to fill 7 of the large, 2-pound jars and 5 of the 9.5-ounce jars. This leaves only 0.5 ounce.

Expressed formally, the problem is to find nonnegative integer values of l (the number of large jars) and s (the number of small jars) so that R (the function that calculates the remaining amount of peanut butter) assumes a minimum nonnegative value. The function R is described here by the mathematical expression

$$R(s, l) = 272 - (32l + 9.5s).$$

In this situation, there is a fairly small number of possible cases to check because the problem only makes sense when R is nonnegative and l and s are nonnegative integers. You can find the answer by exhausting all possible values for l (since its values are larger than those for s). By doing so, you can get an algorithmic feel for the process of choosing values for l (and subsequently, each largest possible s) and calculating the remainders to see how close one can get to 0.

But this can become tedious. Alternatively, you could write a program to compute each remainder for you. Or, by creating a program for a “peanut butter remainder” table, sort of like a multiplication table, you can see many computations at once and scan it quickly for an answer.

A table of “leftovers” is an ideal job for Logo or a spreadsheet. Below is just part of a spreadsheet solution. Select the number of large jars across the top, and the number of small jars down the side. Each entry contains the appropriate value of R , the amount of leftover peanut butter for that particular set of jars.

Large Jars →	0	1	2	3	4	5	6	7	8
Small Jars ↓									
0	272.00	240.00	208.00	176.00	144.00	112.00	80.00	48.00	16.00
1	262.50	230.50	198.50	166.50	134.50	102.50	70.50	38.50	6.50
2	253.00	221.00	189.00	157.00	125.00	93.00	61.00	29.00	−3.00
3	243.50	211.50	179.50	147.50	115.50	83.50	51.50	19.50	−12.50
4	234.00	202.00	170.00	138.00	106.00	74.00	42.00	10.00	−22.00
5	224.50	192.50	160.50	128.50	96.50	64.50	32.50	0.50	−31.50
6	215.00	183.00	151.00	119.00	87.00	55.00	23.00	−9.00	−41.00
7	205.50	173.50	141.50	109.50	77.50	45.50	13.50	−18.50	−50.50
...
14	139.00	107.00	75.00	43.00	11.00	−21.00	−53.00	−85.00	−117.00
15	129.50	97.50	65.50	33.50	1.50	−30.50	−62.50	−94.50	−126.50
16	120.00	88.00	56.00	24.00	−8.00	−40.00	−72.00	−104.00	−136.00
17	110.50	78.50	46.50	14.50	−17.50	−49.50	−81.50	−113.50	−145.50
18	101.00	69.00	37.00	5.00	−27.00	−59.00	−91.00	−123.00	−155.00
...
27	15.50	−17.50	−49.50	−81.50	−113.50	−145.50	−177.50	−209.50	−241.50
28	6.00	−26.00	−58.00	−90.00	−122.00	−154.00	−186.00	−218.00	−250.00

There are many numerical patterns to be explored in such a table, and there are many ways of creating a spreadsheet solution. The important part is understanding how to set up the mathematical expressions necessary to compute the entries.

Problem 2 (*Student page 2*) One approach begins with a qualitative analysis: Where should you go if you have a little bit of money to change? What if you have a lot of money you want to change? Is there a “break-even” point, and if so, where? Once again, a spreadsheet can be used here to obtain an experimental solution. Or, in looking for a break-even point, you could just write an expression for the number of dollars you’d get at each place, set the two expressions equal to each other, and see if there’s a solution that makes sense: Letting ℓ be the number of pounds your friend has,

$$1.65\ell - 2.00 = 1.55\ell.$$

When you solve this equation, you find that, when $\ell = 20$, you’ll receive the same number of dollars no matter whether you go to the bank or the exchange service. If your friend has less than £20, the exchange service is a better deal; more than that, and you’ll find a better deal at the bank.

If you’ve studied functions, however, another approach involves graphing and comparing the functions. We could call them C , for currency exchange, and B , for banking. For example,

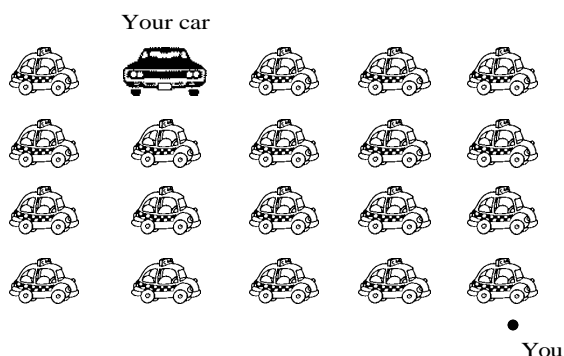
$C(n)$ = the number of pounds you get at the currency
exchange for \$ n .

The bank function would be defined similarly.

MAKING THE LEAST OF A SITUATION

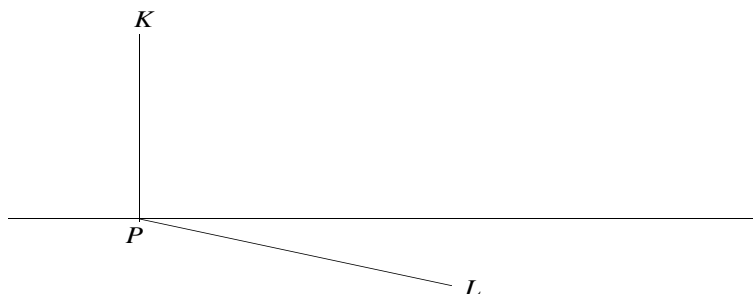
Many famous problems are studied in a system called *taxi-cab geometry*, in which you can move or draw lines only orthogonally (along a grid of lines that meet at right angles).

Problem 1 (*Student page 4*) The problem doesn't specify, but if the cars are parked so tightly that you have to walk along straight, orthogonal lines, then it really makes *no difference* whether you walk along the outside row and turn right at the last column before the car, or whether you turn right immediately, then left, then right, and so on until you reach the car.



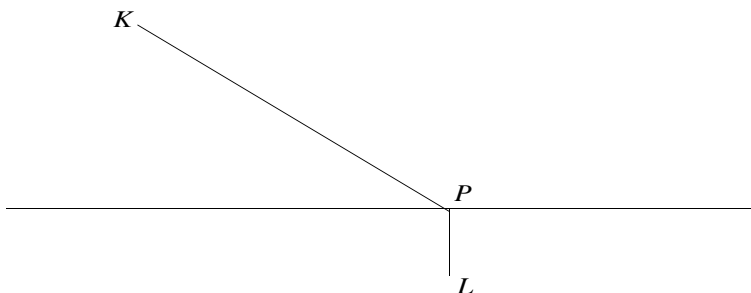
But if there's a fair amount of space between the cars, a better strategy would be to approximate a "straight shot" as best you can by weaving around the cars along a snake-like path of elongated curves. Another strategy would be to take the longest diagonal possible down the closest column in that parking-lot grid, and then follow the longest diagonal from there down the last row before the car.

Problem 2 (*Student page 4*) If you want to minimize the amount of swimming, you should run from L to the point on the shore that's the foot of the perpendicular from K :



The shortest path from a point to a line is along the perpendicular from the point to the line.

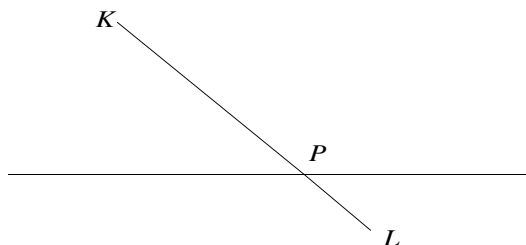
Or, if you want to minimize the amount of running, run straight toward the water:



The shortest path between two points is the segment between them.

Is the location for P in Problem 2c ever halfway between the locations for Problems 2a and 2b?

And if you want to minimize the distance of the total trip, head right towards K :

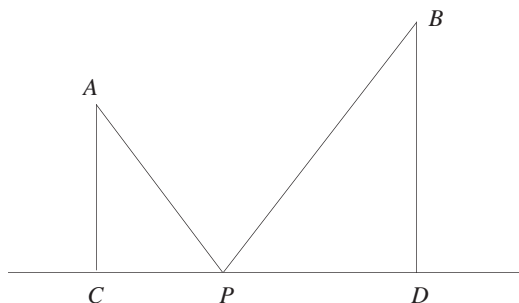


Problems 3–4 (Student pages 6–7) These two problems are meant as opportunities to experiment, to make and test conjectures. In this case, you might use pencil and paper and tools such as a ruler and protractor. Or, you might set up a “river” made of a wooden dowel with a large washer on it to represent movable point P , tie a piece of string to a nail at A , slip it through the washer at P , and pull it taut at B to find the spot for P on the river that results in the shortest distance. Or you might construct the situation with geometry software.

See Problem 8 for a full solution. In brief, to minimize distance, the optimal location for P is at the spot where $m\angle APC = m\angle BPD$, where C and D are at the intersections of the perpendiculars from A and B to the river. (See the figure in the solution for Problem 5.)

Trial-and-error experimentation is a perfectly reasonable way to find a solution to a problem, up to a point. The purpose of doing such experiments is not just to come up with an answer, but to come up with a conjecture you can justify or prove mathematically. Here for example, what do the two congruent angles have to do with minimal total distance from A to the river to B ?

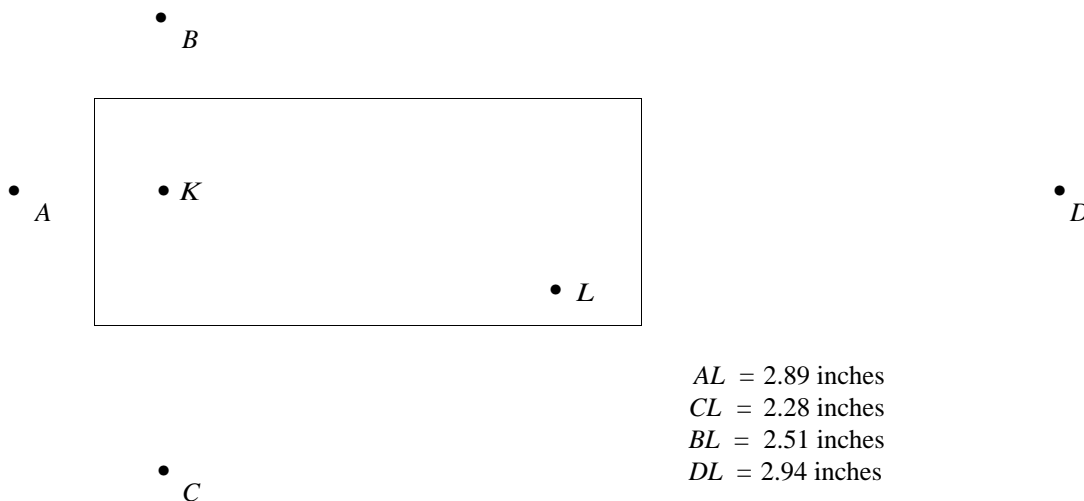
Problem 5 (Student page 7) The distances from A and B to the river (at C and D) are proportional to the distances from A and B to the optimal position for P (as are the distances from P along the riverbank to C and D).



So, when P is at the best spot, $\frac{CP}{PD} = \frac{AC}{BD}$ because $\triangle ACP \sim \triangle BDP$. Therefore, when A and B are the same distance from the river, $AP = BP$. If A is half as far from the river as B , then $AP = \frac{1}{2}BP$.

Problem 6 (Student page 7) As B 's distance from the river increases, the best location for P on the river moves closer to A . (See the solution for Problem 5.)

Problem 7 (Student page 8) The best spot is shown below:



Sometimes this is the way things work in optimization problems: You use theoretical considerations to narrow things down to a few cases, and then you compare these cases with each other by using an explicit calculation.

One way to prove this theorem is to use congruent triangles. Another is to use properties of reflections.

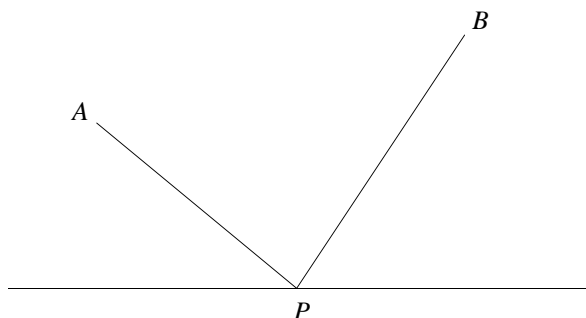
This problem is identical to Problems 3 and 8 except that it also requires choosing the best pool side to use. In the special case when both points K and L happen to be closest to the same side, then clearly that's the side to use. When this is not the case, however, as in the figure in the Student Module, the problem is solved by finding the best place on each of the edges (perhaps using the “reflection technique” described below for Problem 8) and then finding the best place among these four points. (Another approach, using contour lines, will show up later in Investigation 6.4.)

Problem 8 (*Student page 8*) Here we have one of the more well-known versions of a classic minimization problem. Unless the locations for you and the tent (or, for Problem 3, the ship and the refueling station) are the exact same distance from the riverbank, the best place to land is *not* at the midpoint of the segment joining the feet of the perpendiculars from you and the tent. You can reason out where the best spot *must* be by making use of the “perpendicular bisector” theorem (or half of it, as stated here; the other half is its converse):

THEOREM *Perpendicular Bisector Theorem*

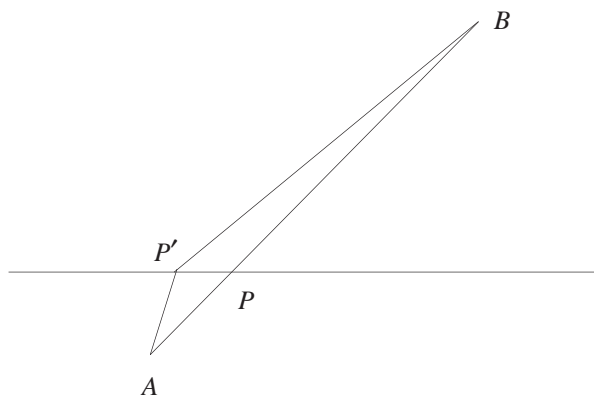
Any point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment.

The problem: I need to minimize the sum $AP + PB$, but P needs to stay on the line.

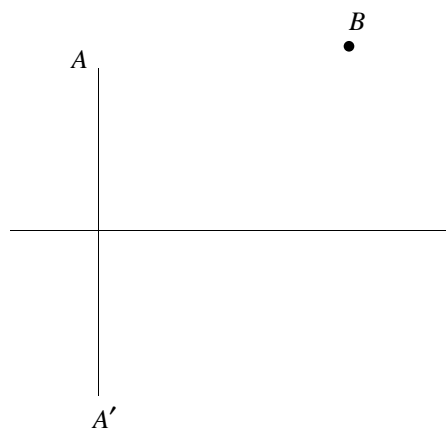


If A were on the other side, this would be just like the run-and-swim problem.

If A were on the *other* side of the line (if A weren't on the same side of the line as B) it would be easy. I could just connect A to B and locate P where \overline{AB} crosses the line of the river. Then any other choice of P , such as P' below, would make a longer trip because the shortest way to get from A to B is along the segment from A to B .



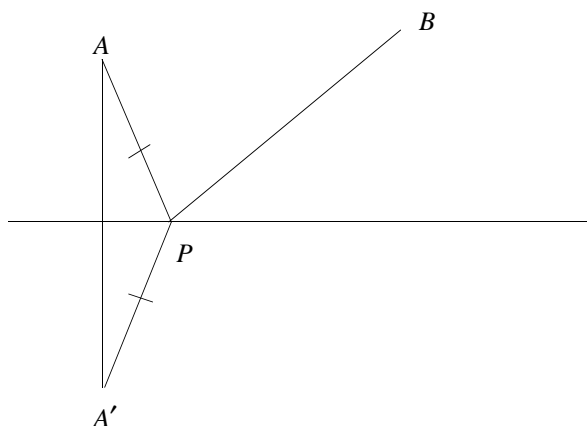
Well, A *isn't* on the other side of the line. But I can take A and reflect it over the line to make A' (so that the line representing the river is the perpendicular bisector of $\overline{AA'}$):



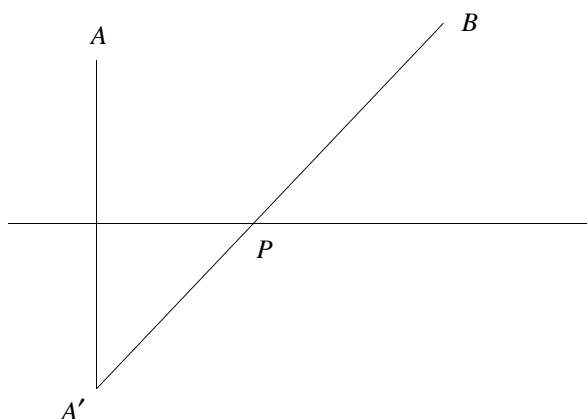
We could call this the "reflection technique."

There's another habit of mind: Often it's possible to change a problem into an equivalent problem that's easier to solve.

Now, going from A to the line to B is the same as going from A' to the line to B , because $AP = A'P$ for any point P on the line:

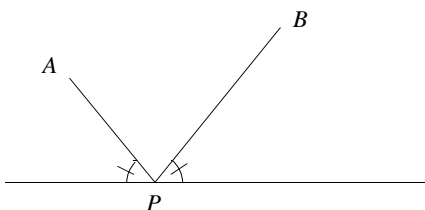


So, just connect A' to B and that's where you should put P :

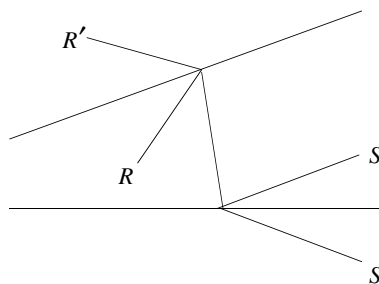


This is how people think of it in optics, where they reflect light rays, and in billiards where they bounce a ball off the edges of a pool table. High school physics students often learn that “the angle of incidence equals the angle of reflection” without having a really clear idea of what that means. Why does this imply that the triangles discussed in the solution for Problem 5 are similar?

Another way to describe the solution is to say that P is the point on the line that makes congruent angles with the shore.



Problem 9 (Student page 8) This problem extends Problem 8 by requiring its methods to be applied twice.



In this picture, R and S have been reflected over the left and right shores. Then any trip

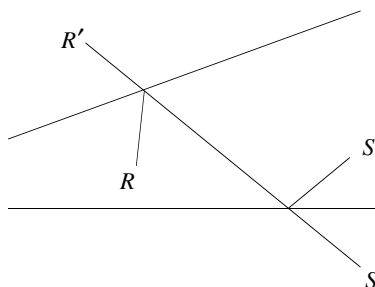
$$R \mapsto \text{Left Shore} \mapsto \text{Right Shore} \mapsto S$$

is the same as

$$R' \mapsto \text{Left Shore} \mapsto \text{Right Shore} \mapsto S'.$$

Can you prove that this is the best path?

But the picture above doesn't show the shortest path. That'll be found by drawing a straight line from R' to S' , and finding where it intersects with the shorelines:



All of the questions in this module assume Euclidean geometry as the context unless specifically noted. For Problem 18 below, we provide some information about how to find the distance between two points on a sphere, but if you'd like to find out more, search for books on other geometries such as elliptic or hyperbolic geometry. One book that's fairly accessible is *The Shape of Space* by Jeffrey Weeks (New York: Marcel Dekker, 1985).

Problem 10 (Student page 9) Here are definitions for the last three terms on the list: Between any two points on the plane, there exist many different paths. A *path* is simply a way to get from point A to point B , and of course, there are many different ways. Each path has a *length* associated to it, measured by various methods, with varying degrees of accuracy. The *distance* between two points is the tricky one—it is defined to be the length of the shortest path between the two points. (Therefore, it doesn't make sense to talk about the “shortest distance” between A and B .)

Problem 11 (Student page 9) In Euclidean space, there is only one shortest path from one point to another: the line segment between the two points. In other geometries lines are defined differently. A perfectly straight line from New York to Anchorage, for example, would pass through the Earth, while the shortest path on the *surface* of the earth (or slightly above, to avoid the bumps) is better approximated by an arc (a segment) of what's called a “great circle” in spherical geometry. (Lines are often called “great circles” in spherical geometry, but the more general term for an intrinsically straight line in *any* geometry is a *geodesic*.)

Problem 12 (Student page 9) The shortest path from a point to a line (in a Euclidean plane) is along the perpendicular to the line.

Problem 13 (Student page 9)

- a. The answer is the same as that for Problem 12—the shortest path is along the perpendicular from A to the line that passes through B and C .
- b. Since the perpendicular does not fall on the segment \overline{BC} , you must take the “best you can get,” that is, the closer of the two endpoints (in this case, B). This idea that the minimum value of a system sometimes occurs at a

boundary point comes up over and over in optimization problems (as well as in calculus).

Problem 14 (*Student page 10*) This must be a *general* strategy—meaning it must take into account *any* situation where you have two points and a line. There are two cases to consider: (1) when the two points are on *opposite* sides of the line; and (2) when the two points are on the *same* side of the line. So suppose the two points are on opposite sides of the line. Then the line segment connecting the two points will do the job.

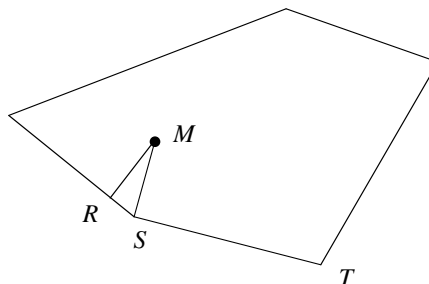
Now suppose the two points are on the same side of the line. In this case, to minimize the total trip from the first point to the line and then to the second point, the general strategy would be to construct the equivalent of a straight segment by using the reflection technique (see the notes for the burning tent problem). By reflecting A over the line and creating A' , you find the spot for P that minimizes total distance at the intersection of $\overline{A'B}$ and the line (because the distance from A to P to B is the same as the distance from A' to P to B).

Problem 15 (*Student page 10*) There are many notable attributes of perpendicular bisectors. A key fact about perpendicular bisectors is the theorem stating that any point on the perpendicular bisector of a line segment is equidistant from the endpoints of the line segment. This is used in the burning tent problem when you reflect point A over the shore line to point A' and state that the distance AP equals the distance $A'P$.

Problem 16 (*Student page 10*) This is another problem that can be solved by first using theory to narrow things down to a few cases and then using explicit calculations to discriminate among the cases. Theory tells us the shortest way out of the pool is along one (or more) of the perpendiculars to the sides of the pool. Experimenting (here that's just measurement and comparison) will tell us which one or more of those perpendiculars is shortest. Whether to swim to a corner or a side, however, is more easily solved theoretically than experimentally. If the swimming pool is convex, the shortest path would always lead you to a side, never a corner. Why?

Try assuming that the shortest path out of the pool *could* take you to a corner; then follow the logic to show a contradiction. So suppose, in that convex swimming pool, that the shortest path to *one* of the sides is actually at a corner of that particular side. (This would be the case when the path to the corner from your spot in the pool at M is perpendicular to one of the sides or is the next best thing because no perpendicular exists from your spot to that side.) Because the pool is convex, the path to that corner

will be the hypotenuse of a right triangle. In the picture below, both angles, $\angle MRS$ and $\angle MST$, are right angles. The distance MS is longer than the distance MR because the hypotenuse of a right triangle will always be longer than either of its legs. Another approach, using contour lines, will be introduced in Investigation 6.4.



Problem 17 (Student page 10) One way is to make the pool a concave polygon. If you insist that the pool is a convex polygon, it can't be done.

Problem 18 (Student page 10) For any two general points on a sphere, the shortest path between them is along an arc of the great circle, or geodesic, that passes through them. *Great circles* are those circles that are as big as possible on the surface of a sphere. Any great circle will define the boundaries of two hemispheres; from the surface of the sphere a great circle will appear to be a straight line. Any other circle on a sphere bends or curves off to one "side" of the sphere.

However, if the two points are on opposite ends of a sphere (if they are *antipodal*, such as the North and South Poles), there will be many shortest paths between them. Points that are exactly halfway around the Earth from each other can be reached equally well in any direction; thus there are an infinite number of equally-minimal distances between two such points.

Problem 19 (Student page 11) There is no minimum value for $SP \times PT$, although it gets smaller and smaller as P gets closer and closer to either S or T . (That smaller and smaller value approaches 0, but when P is actually at one of those endpoints, you can't talk about the angle at P .)

A great circle on a sphere is the intersection of the sphere with a plane that passes through the center of the sphere. You can find the great circle that passes between any two points on a sphere by pulling a piece of string taut between them.

This habit of calculating the same area in more than one way and comparing results shows up quite a bit in geometry.

The maximum for the product occurs when $SP = PT$. Think of it this way: The area of $\triangle SPT$ is half the product of ST and the length of the perpendicular from P to \overline{ST} . So, this area is biggest when the altitude is longest, and that happens when P is “at the top” (that is, when $SP = PT$). But the area is also half the product $SP \times PT$, because the triangle is a right triangle and we can use \overline{SP} as the base. So, $\frac{1}{2}SP \times PT$ is biggest when $SP = PT$, and therefore $SP \times PT$ is biggest when $SP = PT$.

The measure of $\angle P$ never changes; it’s always a right angle. Would the measure of $\angle P$ also be invariant if the measure of arc SPT were 90° , or 45° instead of 180° ?

Problem 20 (Student page 11) Problems similar to Problems 19, 20, and 21 are often used to apply the classical formulas that connect the measures of central angles, inscribed angles, and the corresponding measures of their intercepted arcs. Here they are meant as experiments, to make conjectures and develop ideas or theorems about the relationships between the angles and the arcs they intercept.

Because S and T are fixed, there are only two possible values for $\angle SPT$. The angle measure remains fixed as long as P remains in one of the arcs formed by S and T . If P crosses over S or T into the other arc, then $m\angle SPT$ is 180° minus the first value. To minimize $SP \times PT$, place P at S or T ; to maximize $SP \times PT$, place P across from S and T so that the bisector of $\angle SPT$ intersects the center of the circle.

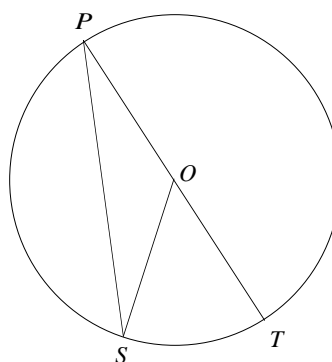
The proof of this fact is similar in spirit to the proof outlined in the solution for Problem 19, but it uses trigonometry. Calculate the area of $\triangle SPT$ in two ways: it’s half the product of ST and the length of the altitude from P to \overline{ST} , and it’s also $\frac{1}{2}PS \times PT \sin P$ where “ $\sin P$ ” means the sine of the measure of $\angle SPT$. But $\sin P$ is constant as P roams around the circle, except at the two points where P meets S or T (and these aren’t allowed, anyway). (This takes some thought: supplementary angles have the same sine, and $m\angle P$ is always one of two supplementary values.) So, $PS \times PT$ is largest precisely when $\frac{1}{2}PS \times PT \sin P$ is largest, and

$$\frac{1}{2}PS \times PT \sin P = \text{the area of } \triangle SPT,$$

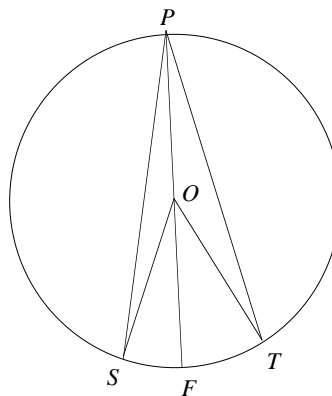
which is largest when P is as far away from \overline{ST} as it can be.

As we've seen in many previous problems, there's an important difference between results that have been found experimentally and results found theoretically. Sometimes there seems to be no choice in the matter; all or part of an answer must be found experimentally, by exhausting all possible cases (for example, Problems 2 and 16 of this investigation). In this problem, however, we can never be really certain of the results described unless a theoretical basis is found. (Otherwise, every circle and angle must be checked.)

One way to show that all the angles inscribed in the same arc have the same measure is to show that the measure of any such angle is half the measure of the associated central angle. You can use what you know about isosceles triangles and angle sums in a triangle to show that the measure of the inscribed angle SPT in the figure below is half the measure of the central angle SOT :



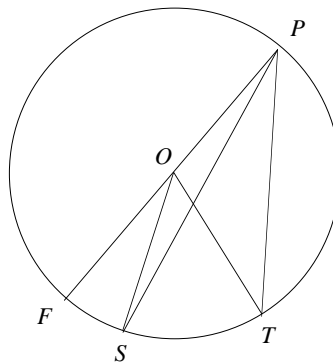
In the figure above, because $m\angle SOP + m\angle SOT = 180^\circ$ and $m\angle SOP + m\angle OPS + m\angle PSO = 180^\circ$, we can conclude that $m\angle SOT = m\angle OPS + m\angle PSO$. Next, since $\triangle POS$ is isosceles, we have that $m\angle OPS = \frac{1}{2}m\angle SOT$. In the figure below, we use two isosceles triangles because P is not on one of the lines containing \overline{OS} or \overline{OT} .



$$m\angle SPF = \frac{1}{2}m\angle SOF$$

$$m\angle TPF = \frac{1}{2}m\angle TOF$$

Now add.



$$m\angle TPF = \frac{1}{2}m\angle TOF$$

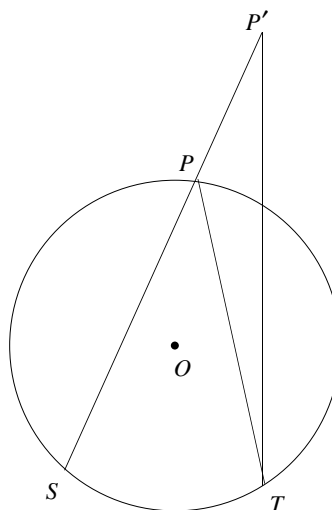
$$m\angle SPF = \frac{1}{2}m\angle SOF$$

Now subtract.

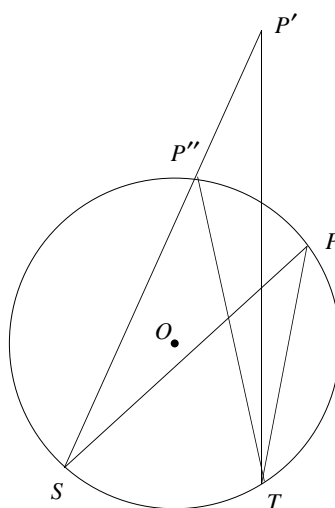
Problem 21 (Student page 12) If P follows a straight path away from S and T as shown in the Student Module, $\angle SPT$ becomes smaller and smaller. Thus, for any point P' outside the circle, $m\angle SP'T$ is always smaller than $m\angle SPT$, where P is on the circle. This result will come up again in Investigation 6.4 on contour lines.

To show that this is indeed always true, there are only two cases to inspect: 1) when P' is outside the circle and collinear with either \overline{SP} or \overline{TP} ; and 2) when P' is outside the circle but not collinear with either of these segments. In the first case, pictured below, $m\angle SP'T$ is less than $m\angle SPT$ because $m\angle PPT + m\angle P'TP = m\angle SPT$.

In other words, $\angle SPT$ is an exterior angle of $\triangle PP'T$.



In the second situation, (below), you can reduce it to a case you already know about:



$$m\angle P = m\angle P''$$

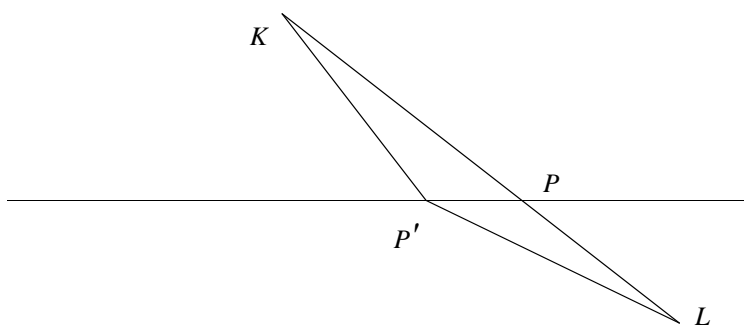
You could summarize all the formulas by the rule “the measure of the angle is half the sum of the measures of the intercepted arcs.” This rule allows for angles that aren’t inscribed in a circle, but then you have to allow for arcs that have negative measure and measure 0.

Mark the intersection of $\overrightarrow{SP'}$ and the circle (call it P''). We know that all angles inscribed in the same arc have the same measure (see Problem 20). This implies that $m\angle SP''T = m\angle SPT$, and the first case examined in *this* problem implies that $m\angle SP'T < m\angle SP''T$.

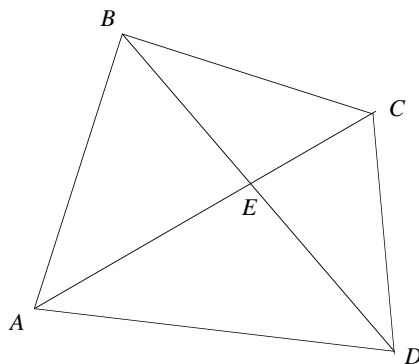
Problem 22 (Student page 12) The Triangle Inequality could be used to justify the answers to Problem 2 (the run-and-swim problem), but, because we already know that the shortest path between two points is defined as the segment between them, and because no proof is provided in the Student Module for the Triangle Inequality, it makes sense to use the definition of the shortest path and the example of the run-and-swim problem to prove the Triangle Inequality:

One student reworded the Triangle Inequality like this: "The length of any one particular side of a triangle will always be less than the sum of the other two lengths." And another wrote: "This theorem is just like saying that two sides of a triangle are one way to get from one point to another, but not the shortest way because they take you to a third place in between. The shortest way is to draw the straight line segment between the two points, which is really just the third side of that triangle."

Let P be a point on any \overline{KL} . Then $LP + PK = LK$, and \overline{LK} is defined as the shortest possible path between L and K . Now consider *any other* point P' not on the line through L and K . Three noncollinear points define a triangle; hence we have $\triangle LP'K$. Now $LP' + P'K$ is another path from L to K , but not the shortest. Thus $LP + PK < LP' + P'K$, meaning that $LK < LP' + P'K$. Therefore, the sum of the lengths of two sides of a triangle are always greater than the length of the third side.



Problem 23 (Student page 12) True; this is one of those inequalities that can be established by writing the Triangle Inequality in as many ways as you can find.



If the quadrilateral is a parallelogram, then the sum of the *squares* of the lengths of the diagonals is *equal* to the sum of the *squares* of the lengths of the sides. Can you write a proof of this?

Are there any locations for the spider and the bug which result in more than one shortest path?

To explain the result that the sum of the lengths of the diagonals is *smaller* than the perimeter of the quadrilateral, use the Triangle Inequality on $\triangle ACB$, $\triangle ACD$, $\triangle BDC$, and $\triangle BDA$ (use each diagonal as the “lonely” third side of a triangle). To find that the sum of the lengths of the diagonals is *bigger than half* the perimeter, use the Triangle Inequality on $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$ (here, use each side of the quadrilateral as the third side of a triangle).

Problem 24 (Student page 13) Because the problem in the Student Module doesn’t specify where the spider and bug are located, there really isn’t any one answer to this problem. Depending on the shape of the room and the two locations under consideration, the shortest path might be for the spider to crawl up to the edge shared with the ceiling, or, it might be that the spider should crawl to an edge shared with either the front or back walls, and from there go on toward the bug. Perhaps the most important part of such a problem is understanding or figuring out a method by which to solve for any particular location.

Consider this method: imagine the room is a box made of paper. Then you could cut or unfold the paper along the edges and lay all the sides of the room out flat. In two dimensions, then, the shortest path will be the distance between the two points. But finding the shortest path may also depend upon comparing two or three different ways to unfold the box: along the top edge, or along the two side edges.

For example, if the spider and the bug were at the centers of their respective walls, the spider would take the shortest path by crawling along the perpendicular to the shared (top) edge. On the other hand, take the case where the spider is on the side wall as pictured in the Student Module, but is also somewhere very close to the shared edge with the front wall, and the bug is on the ceiling anywhere along the shared edge with the front wall. In this case, the Triangle Inequality comes in handy because the *two* shortest paths for the spider approximate 1) the sum of the lengths of two sides of a triangle, and 2) the third side. The shortest path, in this instance, is the second—the diagonal across the front wall.

Problem 25 (Student page 13)

- a. If you have to stay *on* the rectangle, then the only possibility for a shortest path connecting all four points is to use the two shorter sides with one of the longer ones (that is, delete \overline{AB} or \overline{DC} , and add \overline{AD}). This response is perfectly reasonable: you could write a one- or two-line proof of it.
- b. A more complicated situation, however, arises when that restriction is lifted, and you’re allowed to come up with some other network, that is not necessarily on the rectangle. If you experimented with this, you would find there

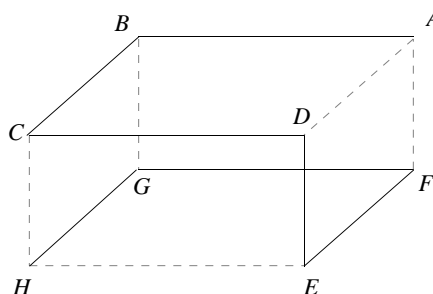
are networks of shorter total length, such as that pictured below. Given that $\angle APD$ and $\angle BQC$ are each 120° , you could use 30–60–90 triangles to prove that the length actually is less than the path on the rectangle described in the first paragraph of this solution.



Problem 26 (Student page 16) To produce a set of paths connecting all of the points, but with shorter total length, remove all vertical segments but the first one, and replace all the horizontal segments with two long horizontal segments, one connecting all the top points, and one connecting all the lower points.

Problem 27 (Student page 16)

- a. The solution shown in the Student Module ($ABCHGFED$) makes use of two short lengths (the heights \overline{CH} and \overline{ED}) and five long ones. There are a number of better solutions. One, which uses three short segments, is $ABGFEDCH$.
- b. A worse solution is one that uses only one short segment, like that pictured below.

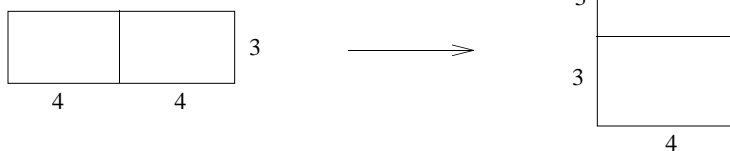


- c. There are eight vertices, so there have to be seven connecting segments. The best we could hope for is to look for a path that uses all four short segments, and three of the long ones. One such path is $AFEDCHGB$.

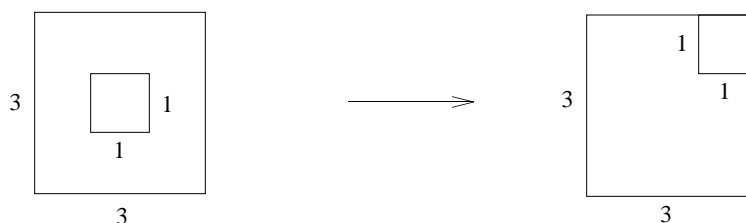
- d. The shortest noncontinuous path would still only have four of the short segments, since there aren't any others. One example is to connect $EFGH$ with the four perpendicular segments at those vertices. The sum of those seven lengths is the same as the best solution for part c. However, a good question to ask, if travel is allowed in any direction, is whether this noncontinuous path provides for a total of less travel when you consider the sum of all paths from any one of the vertices to any one of the others.
- e. If the box has a nonsquare rectangular base, assuming the shorter side of the base is not exactly the same length as the height of the box, then you have a third set of segments to consider. It doesn't matter whether the shorter side of the base is longer or shorter than the height, because there are now three sets of four equal lengths. You need seven to connect all the vertices, so you find a way to use four of the shortest lengths, two of the medium lengths, and one of the long ones.

Problem 28 (Student page 17) Here are some solutions. (They aren't the only ones.)

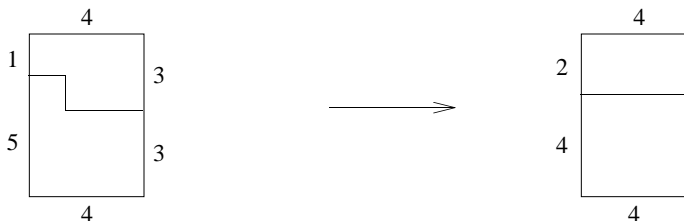
a.



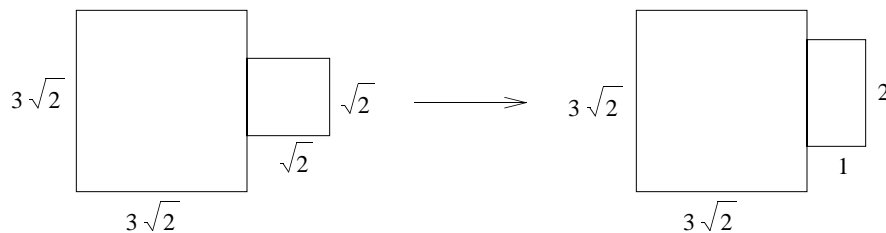
b.



c.



d.

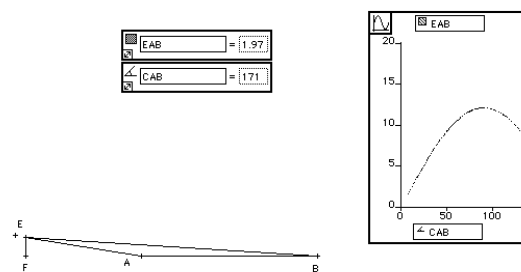
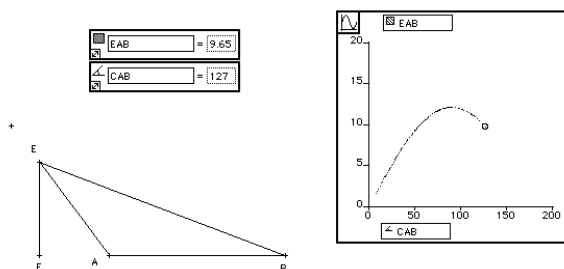
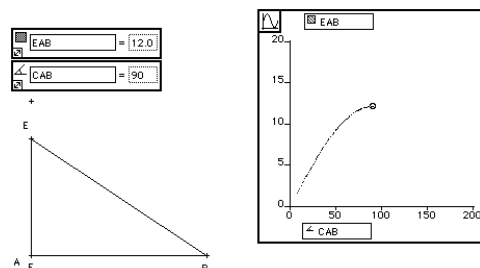
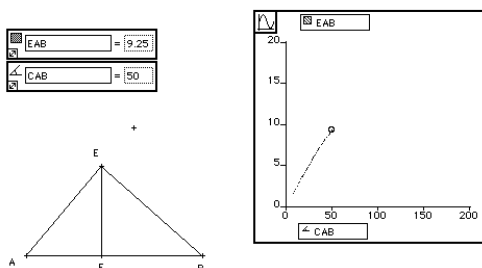
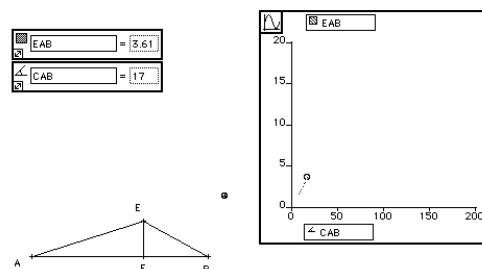
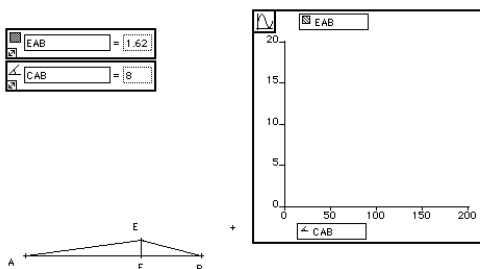


There are some general strategies here. For example, “zig-zags” can be replaced with straight line segments to save an entire horizontal or vertical. More generally, savings can be realized whenever the pens can be made to share an unshared side, or to lengthen a previously-shared side. The last example is a foreshadowing of the upcoming “pen against the wall” problem (see Problem 6 of Investigation 6.3 in the Student Module).

MAKING THE MOST OF A SITUATION

Here we use reasoning by continuity again.

Problems 1–2 (Student page 20) Of all triangles with sidelengths a and b , the one with the greatest area is the one with a right angle included between a and b . To see this, imagine a dynamic experiment starting out with the sides close together. In the figures below, created by the software Geometry Inventor™, let \overline{AB} represent the side of length a , and \overline{AE} represent the side of length b :



Conversely, you might see *this* problem as the “easier” of the two, and convince yourself it’s the right triangle that maximizes area because the right triangle is half of a rectangle.

This is a classic optimization problem in which “best” means “greatest area.”

Some methods yield a precise answer, others may yield a conjecture to motivate the search for a reason, a proof, or precision.

The dynamic experiment helps to visualize what we might deduce from thinking about how area is calculated: that the area of the triangle will be half the product of one of the given sides and the height, and the height is at a maximum when it coincides with the other given side. (That is, when the given sides a and b are the height and base of the triangle, and the included angle, therefore, is 90° .)

Problem 2 can be solved by thinking of the parallelogram as made up of two congruent triangles and reducing the situation to the one in Problem 1. Because the right triangle maximizes area, of all parallelograms of sidelengths a and b , it is the rectangle that will have maximum area ab .

Problem 3 (Student page 21) The best rectangle for a given perimeter is a square. There are several ways to investigate this, including experimental, numeric, algebraic, geometric, and analytic techniques:

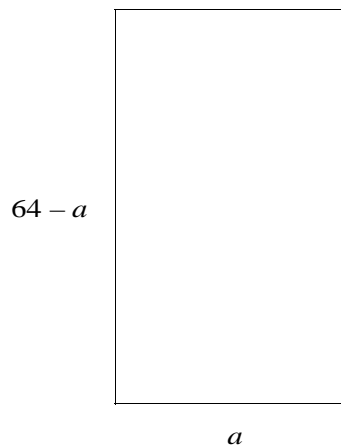
- Use some string of fixed length (tie it together at, say, 128 cm or inches, and then experiment with other lengths). With a partner, look at different rectangular shapes and their respective areas, or set up a sketch using geometry software.
- Because the perimeter of any rectangle is twice the sum of the lengths of two of its nonparallel sides, look at pairs of numbers whose sum is 64 and that yield the greatest product. (Some of these pairs are “symmetric,” like $(32 + 1)(32 - 1)$, $(32 + 2)(32 - 2)$, $(32 + 3)(32 - 3)$, and so on. Or, for a more theoretical approach, represent any sum with $(x + a)(x - a)$, $(x + b)(x - b)$, $(x + c)(x - c)$, and so on.)
- Set up an algebraic expression which gives the area of the rectangle as a function of one dimension (the perimeter). Then graph the resulting expression and locate its maximum either by estimation or by analytic techniques.
- Work through a geometric proof of the fact that the square is the best rectangle for a given perimeter. At least try this before working on Problem 4. Such a proof is provided in Problem 4 of the Student Module because an important part of doing mathematics is to be able to read and make sense of another person’s argument.

Problem 4 (Student page 21)

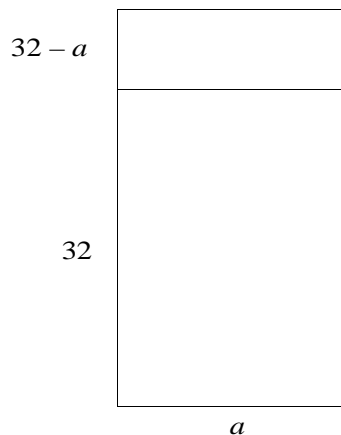
- c. One student’s answer goes like this: “I’ll show that a 32×32 square is best by showing that any other rectangle with perimeter 128 has area smaller than the area of the square. I’ll do it by showing I can cut up such a rectangle and make it fit inside the square with some pieces left over.

If the rectangle isn't a square, one of its dimensions has to be smaller than 32.

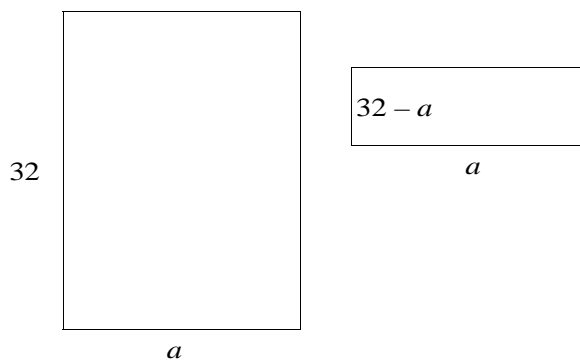
“Suppose, for example, the base of the rectangle is a , and a is smaller than 32.



“First, I'd cut it like this:

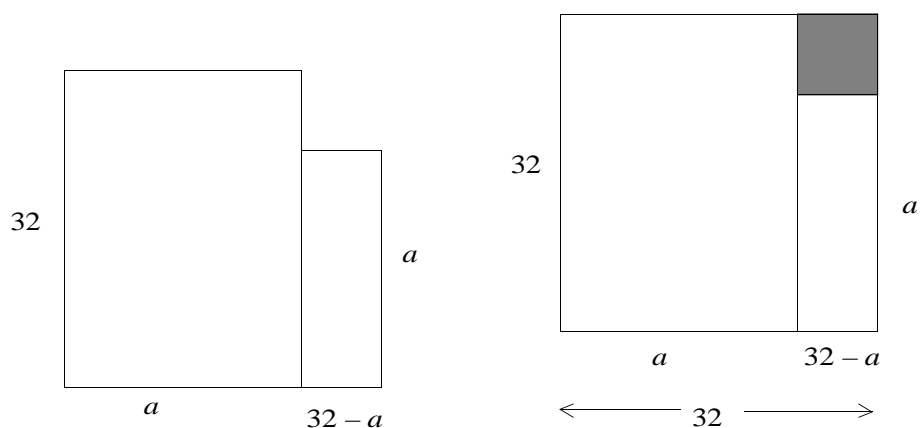


“Then, I take the top strip off like this:



This picture makes use of the fact that $a < 32$.

“And I put it beside the $a \times 32$ rectangle:



“This operation *rearranged* the area, but I didn’t add or lose any. Now, my two pieces cover some of the square, but not all of it. So the square definitely has more area than any nonsquare rectangle of the same perimeter!”

Problem 5 (Student page 22) This is a problem that has many good solutions. Here are two solutions:

- Go through a construction like the one outlined in the solution for Problem 4, but this time do it in full generality. Suppose the rectangle has dimensions $a \times b$ with $a < b$. The side of the square in this case is $\frac{a+b}{2}$. Follow the argument presented in the solution for Problem 4, replacing 64 by $\frac{a+b}{2}$. This will work.
- If p is the perimeter of the rectangle with sides of length a and b , then $p = 2(a+b)$. The side of the square with the same perimeter is $\frac{p}{4}$. Now, $a + b = \frac{p}{2}$ implies

$$b - \frac{p}{4} = \frac{p}{4} - a.$$

(If $a < b$, both sides of this equation are positive.) Let $c = b - \frac{p}{4}$, then also $c = \frac{p}{4} - a$. Therefore,

$$\begin{aligned} \text{area of rectangle} &= ab \\ &= \left(\frac{p}{4} - c\right)\left(\frac{p}{4} + c\right) \\ &= \left(\frac{p}{4}\right)^2 - c^2 \\ &= \text{area of square} - c^2 \\ &< \text{area of square.} \end{aligned}$$

The area of the shaded region in the figure on page 25 is exactly c^2 .

Could the geometric mean ever *equal* the arithmetic mean?

What would be a good definition for geometric mean of *three* numbers?

Another proof uses a result that comes up over and over in mathematics: the *arithmetic-geometric mean inequality*. It says simply that if a and b are positive real numbers, then

$$\sqrt{ab} \leq \frac{a+b}{2}$$

The arithmetic mean of two numbers a and b is their average. It is the number m that makes the sequence

$$a, m, b$$

an *arithmetic* sequence ($m - a = b - m$). The geometric mean of two numbers a and b is the number r that makes the sequence

$$a, r, b$$

a *geometric* sequence (so that $\frac{a}{r} = \frac{r}{b}$). The arithmetic-geometric mean inequality says that the geometric mean is never bigger than the arithmetic mean. To establish the arithmetic-geometric mean inequality, argue like this: If a and b are real numbers, then

$$(a - b)^2 \geq 0.$$

because the square of any number is never negative. Thus,

$$a^2 - 2ab + b^2 \geq 0,$$

and so

$$a^2 + 2ab + b^2 \geq 4ab.$$

(This comes from adding $4ab$ to both sides.) Therefore,

$$\frac{(a+b)^2}{4} \geq ab.$$

Since both sides are nonnegative, we can take square roots, yielding the arithmetic-geometric mean inequality.

Now, back at the rectangle problem:

$$\begin{aligned} \text{area of rectangle} &= ab \\ &\leq \left(\frac{a+b}{2}\right)^2 \\ &= \text{area of square,} \end{aligned}$$

where the inequality is obtained by squaring both sides of the arithmetic-geometric mean inequality.

Problem 6 (Student page 22) With a few modifications, Problem 6 can be approached by the methods discussed for Problem 3. One student reasoned through it like this: “Imagine the pen is first made freestanding. To maximize the area for the given perimeter of 32 meters, you make it a square. It has an area of 64 square meters. Now consider the restriction that this pen must be constructed along one side of the barn. You have to take away one whole side of the square pen (8 meters long) and redistribute it among the remaining three sides in such a way that you still have a rectangular pen, but also have maximum possible area.

“Well, if you redistribute any part of the 8 meters to a side that is perpendicular to the barn, then you have to redistribute that same amount to the other side perpendicular to the barn in order to retain the desired rectangular shape. Say you add 4 meters to each, this adds a total area of 8×4 or 32 square meters. But if instead you add all of the 8 meters to the one side that’s parallel to the barn, then you just shift the sides that are perpendicular to the barn, and your area grows by a full 8×8 , or 64 square meters. Anything else you can do would be something in between these two possibilities—it would involve adding some to the side that’s parallel to the barn and some to the two perpendicular sides, and every time you add length to the two perpendicular sides you have to use some of your 8 meters on *both* of these sides to retain the rectangular shape. So it’s most efficient to add it to the side that’s parallel to the barn.”

Another way to look at this problem is to consider it as a variant of the “reflection technique” (see the solution for Problems 3–9 from Investigation 6.2), which might be better renamed here the “symmetry technique.”

For example, for any pen of perimeter 32 meters with one side against the barn, you can imagine a pen of perimeter 64 meters made completely out of fence, by thinking of the union of the 32-meter pen you already have and its reflection image over the barn’s wall (and suspending, temporarily, the barn’s existence).

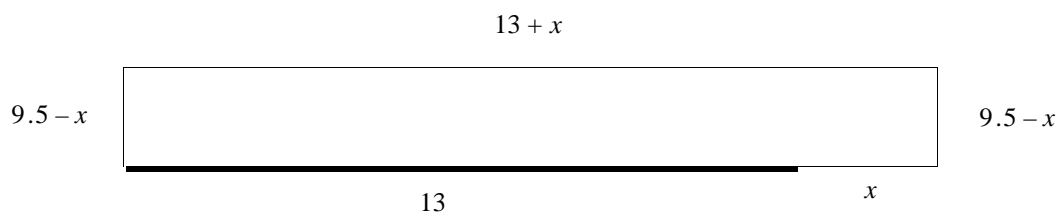


The largest-area pen with a perimeter of 32 meters will occur when you have the largest-area pen for a perimeter of 64 meters (because the area of the 32-meter pen is half the area of the 64-meter pen). But, by Problem 3, the 64-meter pen has maximum area when it’s a square; that is, when it has sides of length 16. Therefore, the pen against the wall has maximum area when its dimensions are 8×16 .

Problem 7 (*Student page 22*) This is one of those problems where endpoints come into play. If your barn were long enough, you’d use 16 meters of it for a side of the pen and have an area of 128 square meters for 32 meters of fencing (see the solution for Problem 6). But it’s not that long, so in this case you’ll build a pen that’s 13×9.5 , for an area of 123.5 square meters, again using your 32 meters of fencing.

One student had the clever idea that maybe we aren't covering all the bases with this argument:

“What if you use a bit of the fence to extend the wall?”



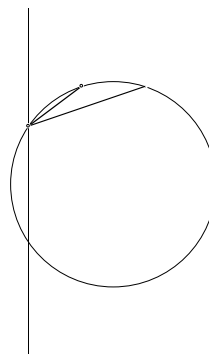
Maybe you can pick up more area that way.” In the picture above, the wall has been extended by some positive number x , so the other lengths are as marked (remember, you only have 32 feet of fence). Then the area of the pen is

$$(9.5 - x)(13 + x) = 123.5 - 3.5x - x^2 = 123.5 - (3.5x + x^2).$$

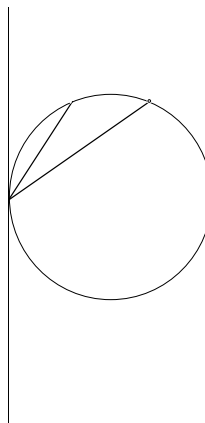
But, if x is positive, so is $3.5x + x^2$, so this area will always be 123.5 minus something, which is less than what we can make the area by using all the fence on three sides.

It was a good idea, though.

Problems 8–9 (Student page 23) Geometry software allows you to experiment with these problems. If you want to maximize the angle from one goal post to you to the other goal post, you might pass a circle through the goals and you:



Then the angle is largest when the circle is tangent to the kicking line:



This is because any *other* point on the kicking line is *outside* the tangent circle, so it makes an angle that is smaller than the one formed at the point of contact (see the solutions for Problems 19–21 from Investigation 6.2). In fact, this argument shows where to kick the ball if you are running along *any* line that doesn't intersect the segment between the goals (solution to Problem 9).

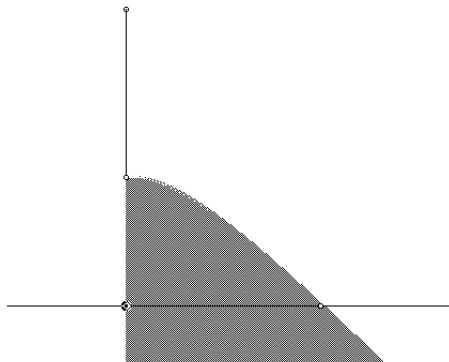
For Problem 9, the following are a few other examples:

The art gallery problem: You are in a gallery looking at a picture which is hanging in front of you. Your eye level is 5 feet, the bottom of the picture is 6 feet above the ground and the picture is 4 feet tall. Where should you stand so that the angle from the bottom of the picture to your eye to the top of the picture is a maximum?

The hockey problem: A hockey player skates toward the goal net along a line perpendicular to the net (but off to the side, so that the skating path doesn't intersect the segment between the ends of the net). What is the best place from which to take a shot?

Jon Choate, a teacher at Groton School in Connecticut who also coaches hockey, says that players develop an intuition for finding the place that maximizes the angle between the puck and the goal ends. Jon has thought about the following situation:

When players warm up, they skate out on various lines that are perpendicular to the goal. If each player shoots from his or her “best” spot, what is the locus of these shooting positions?



Can you describe the curve? The area “beneath” this shooting curve is referred to as the “gray area” by hockey enthusiasts.

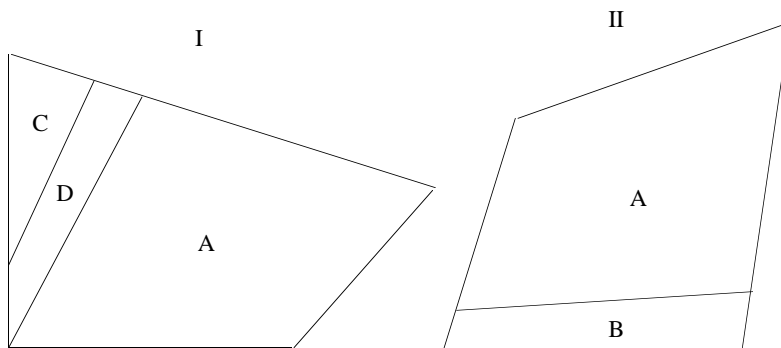
Problem 10 (Student page 24) A square is the rectangle of greatest area for a given perimeter.

Problem 11 (Student page 24) There are several strategies here, all of them approximate. For example, you can:

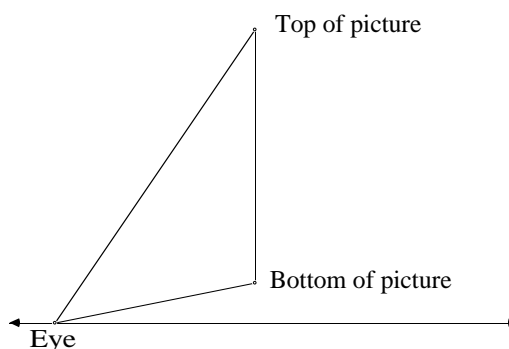
- Triangulate each figure and approximate its area.
- Cut one figure up and show that it fits inside the other with some left over.

Here’s an example of the second strategy mentioned above:

Call the polygons I and II. We’ll show that I is the larger of the two. Notice that, if we draw a line segment through II and call the resulting pieces A and B, piece A also fits inside polygon I as shown below. In fact, I can be divided into piece A and two other pieces; call them C and D. Thus $I = A + C + D$ and $II = A + B$. Again by cutting and comparing, we can see that piece D fits inside B, but there is not enough room inside B for both D and C. This means $D + C$ is larger than B, implying that $A + D + C$ is larger than $A + B$, so I is larger than II.



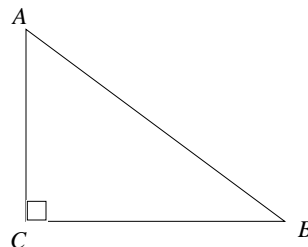
Problem 12 (Student page 24) Assuming that you are standing “head on” to the picture (so that you are somewhere on the plane that is the perpendicular bisector of the bottom edge of the picture), this is another version of the soccer problem. Here’s a side view of the situation:



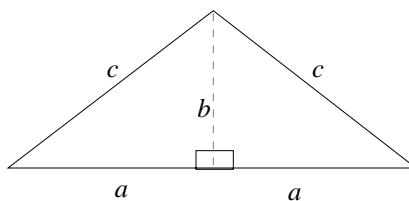
You want to maximize the angle from the top of the picture to your eye to the bottom of the picture. In this case (head-on in front of the picture), there is one place along the line that represents the possible positions for your eye where the angle is inscribed in a circle tangent to that line. That location offers the maximum viewing angle.

Problem 13 (Student page 24) The larger angle is the one inscribed in the smaller circle, because the vertex of the angle inscribed in the larger circle is *outside* the smaller circle (see Problems 19–21 from Investigation 6.2.)

Problem 14 (Student page 24) There are a lot of ways to explain why the hypotenuse is the longest side of a right triangle. One reason is that $\overline{AC} \perp \overline{BC}$. The perpendicular is the shortest distance from a point to a line, so any other path from A to \overline{BC} (like \overline{AB}) has to be longer.



Another way to get this result is by using the Triangle Inequality. Suppose that we have a right triangle with sides of length a , b , and c , where c is the length of the hypotenuse. Place two copies of the triangle together like this:



Notice that we do indeed get a new triangle (adjoining the two 90° angles produces a straight line at the base), which has sides of lengths c , c , and $2a$. By the Triangle Inequality,

$$c + c > a + a$$

$$2c > 2a$$

$$c > a.$$

This shows that the hypotenuse is longer than the side of length a , but the same argument with a replaced by b will show that the hypotenuse is also longer than the side of length b ; hence it's the longest side.

A third and simple way to show that the hypotenuse is the longest side of a right triangle is to apply the following theorem:

THEOREM

If two sides of a triangle are not congruent, the angles opposite them are not congruent, and the larger angle is opposite the longer side.

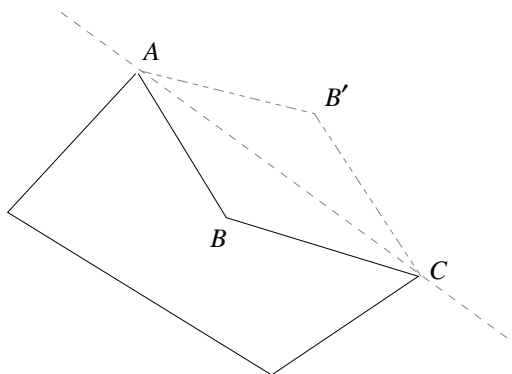
The right angle is the largest angle in any right triangle, so the side opposite the right angle, which is the hypotenuse, must be the longest side.

But understand that this only produces experimental evidence.

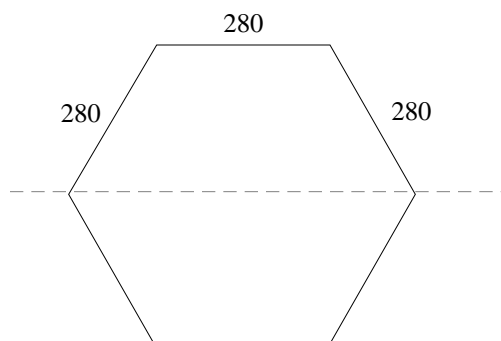
Problem 16 (Student page 25) Here we have compiled a list of strategies that have come up so far in solving optimization problems. Yours may be different.

- When working with area, use the fact that given two shapes A and B , if shape A can be cut and rearranged into pieces which fit inside shape B then A has smaller area than B . (See Problems 4, 5, and 11.)
- Use geometry software to see all possibilities for a certain situation, and then decide when the situation is optimal. (See Problems 1 and 2.)
- Use the symmetry principle (see Problem 6). Sometimes it's easiest to maximize or minimize two times (or one half times) a certain quantity instead of the quantity itself.
- Use the reflection principle. (See the burning tent problem, which is Problem 8 of Investigation 6.2.)
- When working with angles, relate the size of an angle to the position of its vertex relative to the circle (inside, on, or outside the circle \rightarrow bigger, smaller, smallest).
- Use the Triangle Inequality. (See Problems 22 and 23 of Investigation 6.2, and Problem 14 of this investigation.)
- Use the fact that the shortest path between two points in the plane is a straight line segment, or that the shortest path from a point to a line is the perpendicular from that point to the line. (See Problem 2 of Investigation 6.2.)

Problem 17 (Student page 25) The given polygon has only one concavity, and it can be removed in a way that keeps the perimeter constant and increases the area. Suppose that A , B , and C are three consecutive vertices of a part that “sticks in.” Then reflect \overline{AB} and \overline{BC} over \overline{AC} , keeping the perimeter the same and increasing the area.

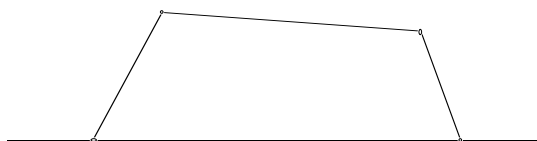


Problems 18–19 (Student page 26) In Problem 18, the greatest area is enclosed by half of a regular hexagon of side 280:



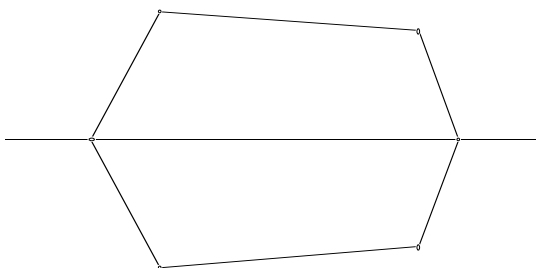
How do we know this shape maximizes area? Theorem 6.2 says that a regular polygon maximizes area for a given perimeter. Problem 18 says that you want to build a pen in the shape of a quadrilateral—any kind of quadrilateral, as long as you maximize area. Assuming you worked through Problem 6, you know that a rectangle with sides in the ratio of 2:1 maximizes area for a rectangular pen against a wall. Such a pen is half a square—the regular polygon of four sides. In this case, however, your pen needn’t be a rectangle. You can obtain more area by thinking of swinging out the two sides that are perpendicular to the wall. (Use the vertices these sides share with the length of the pen parallel to the wall as hinges—this is similar to the strategy just used in Problem 17.) Now you have a trapezoid of slightly greater area than the half-of-a-square rectangle from Problem 6. Adjusting your trapezoid so that the three sides created by fencing are all of the same length creates half of a regular hexagon and maximizes the area.

Let’s make this precise by using the “symmetry technique.” Suppose you were free to make this pen against the wall with *any* three sides that add up to 840:



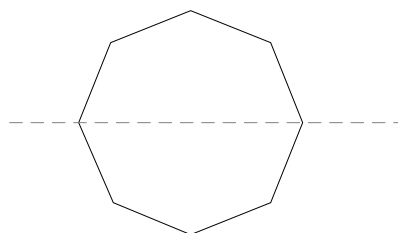
The symmetry technique for the rectangle against the barn is described in the solution for Problem 6.

Then you could reflect the pen over the wall to get a hexagon with twice the perimeter and twice the area:

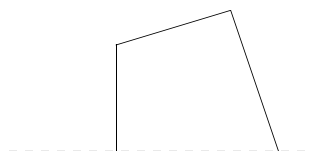


Maximizing the pen's area is the same as maximizing the hexagon's area, and that happens when the hexagon is regular. So, the best pen is half a regular hexagon.

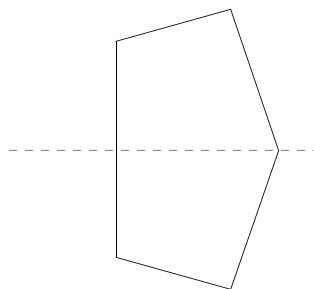
Similarly in Problem 19, the best five-sided pen with one side made from an existing wall is half a regular octagon.



There is, however, a subtle flaw in our argument. Suppose, for the four-sided figure in Problem 18, you happened to start with a figure like this:



Then when you reflect it, you end up with a pentagon, rather than a hexagon:



A regular pentagon maximizes the area for this case.

These polygons are built from the respective pens and their reflections over the wall. So all these polygons have perimeter 1680.

We got these approximations using the formula $A = \frac{1}{2}Pa$ for the area of a regular polygon, where P is the perimeter and a is the apothem (the distance from the center of the polygon to one side). In most cases, a has to be approximated using trigonometry. See the *Connected Geometry* module *A Matter of Scale* for more details.

What's worse is that you could have started with two sides of the fence perpendicular to the wall, so that when you reflect it you would get a quadrilateral.

So, in Problem 18, we're faced with comparing the areas of three polygons:

- a square with side 420;
- a regular pentagon with side 336;
- a regular hexagon of side 280.

The actual candidates for the *pens* we are after are each “half” of these figures (the half that lies on one side of the wall), but if we maximize the area of the whole polygon, we maximize the area of the pen. Whether the answer is found by approximation, or by using area formulas, the hexagon has greatest area. Our approximations show that the area of the square is 176,400, the area of the pentagon is about 194,200, and the area of the hexagon is about 203,700.

Similarly, in Problem 19, there are three cases to consider:

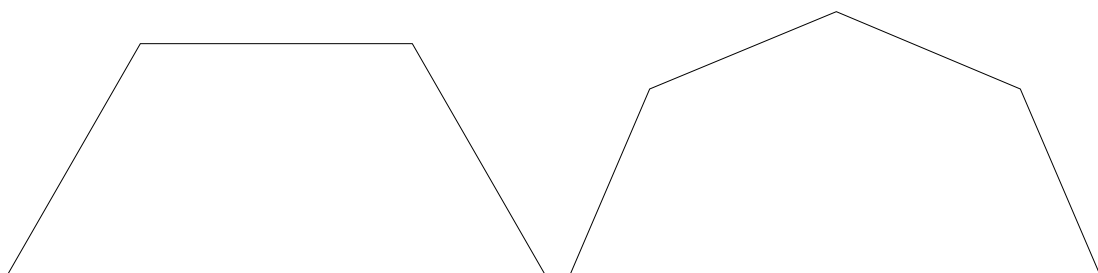
- a regular hexagon with side 280;
- a regular 7-sided polygon with side 240;
- a regular octagon of side 210.

Approximation (or formulas) will show that the octagon wins (its area is about 212,900). We'll leave the area of the 7-gon for your enjoyment.

Problem 20 (*Student page 26*) Will the regular polygon (of a given perimeter) with the largest number of sides always win at this game? Your conjectures in this matter foreshadow some of the work you may do in the last section of this module, “The Most Area.”

This problem asks you to compare the areas of half a regular octagon of side 30 and half a regular hexagon of side 40. If you know some trigonometry, this can be a very nice calculation (the half-hexagon is made up of three equilateral triangles of side 40, and the half-octagon is made up of 4 isosceles triangles, each with base 30 and a 45° vertex angle). If you don’t know the necessary trigonometry, you might do a cutting experiment.

Be careful: The “bases” of the two figures look the same length, but they aren’t quite equal.



Note the difference between this problem, (one side and the sum of the other two are fixed), and Problem 1 of this Investigation (two individual sidelengths are fixed and the third is free to be as short or long as possible). In Problem 1, the right triangle maximizes area.

Problem 21 (*Student page 26*) The area of a triangle is half the product of its base and height. If the base is held constant, the area will be as large as possible when the height is as large as possible. That’s what’s happening here: The base, \overline{AB} , remains constant as C moves around, so the area of the triangle will be largest when the height is largest. The largest possible height for this triangle happens along the perpendicular bisector of the base, when $\triangle ABC$ is an isosceles triangle.

Problem 22 (*Student page 27*) The resulting curve is an ellipse. Ellipses will also come up in a central way, as contour lines, in Investigation 6.4.

Another way to define an ellipse is as the curve you get when you slice a cone with a plane in a certain way. An ellipse is one of the curves called a “conic section.” These two definitions are essentially the same.

Given a rectangle of fixed perimeter, the square provides maximum area; given a rectangle of fixed area, the square provides minimum perimeter.

Looking at data is a good method to use to begin to attack a problem such as this one.

One way to *define* an ellipse is as the set of all points where the sum of the distances from those points to two fixed points is a constant value. The two fixed points are called the *foci* of the ellipse.

Problem 23 (Student page 27) Theorem 6.2 states that the triangle with the greatest area will be a regular triangle, which means in this case an equilateral triangle with sidelength 8. Note that the theorem specifies a fixed perimeter, but no particular sidelength is fixed. (Note, however, that this theorem was never proved. Try solving the problem using a few applications of the ideas in Problem 21.)

Problem 24 (Student page 27) The logic here is a bit subtle. The answer is that the square (with sides equal to 8) will have the smallest perimeter of any rectangle with fixed area (in this case 64). Below are a couple of solutions provided by students and teachers.

- Take a rectangle with sides of length a and b and area 64. We want to minimize the perimeter, $2(a + b)$, so we might as well just minimize the quantity $a + b$. By making a chart of different values of $a + b$ (with the given condition that $ab = 64$), you find that $a + b$ is smallest when $a = b = 8$.
- More precisely, we can use the *arithmetic-geometric mean inequality*, discussed in detail in Investigation 4.16 in the *Connected Geometry* module *A Matter of Scale*. This inequality says that if a and b are positive numbers, then

$$\sqrt{ab} \leq \frac{a + b}{2},$$

and the two sides are equal precisely when $a = b$. Now, if the sides of our rectangle are a and b , its area is ab ; so ab is *constant* (that’s what we are assuming: constant area). This implies that \sqrt{ab} is also constant; call it K . Then the perimeter of our rectangle, $2(a + b)$, is always bigger than $4K$ (multiply both sides of the arithmetic-geometric mean inequality by 4). And, in fact, $2(a + b)$ is equal to this smallest possible value precisely when $a = b$; that is, precisely when our rectangle is a square.

- We want to make the sum $a + b$ small. But if we decrease a too much, then b will have to be pretty large to make sure that $a \times b = 64$. For example, if $a = 1$, then we must have $b = 64$; then $a + b = 65$, which is quite large. So we want both a and b to be relatively small, and close to each other in size. This might lead us to guess (and then check) that $a = b = 8$ is what we want.

This particular technique, using proof by contradiction, will be seen again later, in the last section of this module. Notice how it assumes that there *is* a best possible rectangle. How can we be sure of that?

- We want to show that, of all rectangles with area 64, the square has the smallest perimeter. Suppose that R is the rectangle of area 64 with the smallest perimeter. Let's call its perimeter k . Now let S be the square whose perimeter is also k , in other words, the square with sides of length $\frac{k}{4}$. Since S and R are both rectangles with the same perimeter, we know that S has greater area because it's a square (recall Problem 4). Therefore,

$$\text{Area}(S) > \text{Area}(R)$$

$$\frac{k^2}{16} > 64$$

$$k^2 > 1024$$

$$k > 32.$$

This means that R , the rectangle of area 64 with smallest perimeter, has a perimeter *larger* than 32. But the square of area 64 has perimeter *exactly* equal to 32, contradicting that R 's perimeter is supposed to be smallest! The only way this could happen is if R and S are equal, that is, if the square has the smallest perimeter.

CONTOUR LINES AND
CONTOUR PLOTS

Problem 2 (*Student page 29*) It might be simpler to make a contour plot of an outdoor amphitheater than of a gym with all the bleachers pulled out. Amphitheaters usually have simple staircase-like platforms for people to sit on, while bleachers have seats as well as footrests in between each row of seats. If you make a contour plot of a gym, you might choose to represent only the seats.

There is an important distinction between the contour plot for something like bleachers or a staircase and the contour plot for something like a hillside—the bleachers rise in discrete steps, whereas a hill or mountain peak most often exhibit a more continuous change in height. In later problems, you’ll see how a surface plot would reveal these differences better than a contour plot.

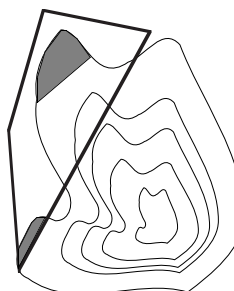
Problem 3 (*Student page 29*)

- a. If the outermost contour line represents 100 feet above sea level, then one place where the mountain will be approximately 1600 feet is halfway between the second and third contour lines. Of course, in a few places between the third and fourth lines, the mountain *might* dip down again to 1600 and then rise steeply to 3100 feet, but we can be sure it doesn’t rise to 1600 feet between the first contour line, which represents 100 feet above sea level, and the second, which represents the place where the mountain has first risen to 1100 feet.
- b. The contour lines show 100, 1100, 2100, 3100, and 4100 feet, respectively. As the highest contour shows 4100 feet, the peak is greater than that but less than 5100 feet.
- c. As you approach the peak from the “south,” you pass contour lines packed more closely together than anywhere else (although one particular spot in the southwest comes pretty close). Said in another way, more vertical distance (number of contour lines) is passed for a given distance toward the peak than elsewhere. This is especially true in the middle of that journey, where the contour lines are packed most closely.

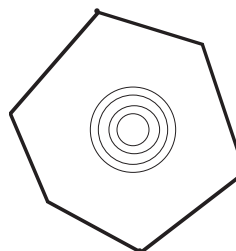
Problem 4 (Student page 29) The dot marks the highest point on the path, roughly 3100 feet.



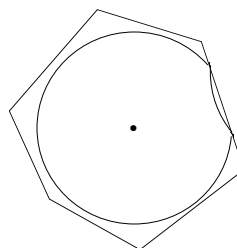
Problem 5 (Student page 29) Assuming the outer contour marks the edge of the lake, there are many possible places. The lower shaded region is more private—away from other swimmers. The upper region keeps the children in view of everyone’s watchful eyes and provides them a larger swimming area. The upper shaded region only represents part of the region where the water is 5 feet or less.



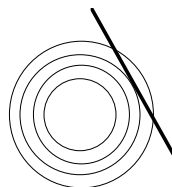
Problems 6–7 (Student page 30) Each circle is a curve of constant distance from you. When any ripple first touches a side of the pool, all the rest of that ripple is within the pool. Swimming to the point of tangency gets you to a side; swimming to any other point on that circle leaves you farther to go.



Problem 8 (Student page 31) The part of the ripple that would have expanded beyond the wall will travel back toward the center of its origin, as if reflected over a tangent line.



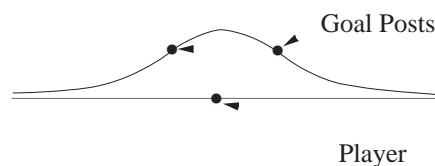
Problem 9 (Student page 31) After a ripple has expanded beyond tangency to the wall, it touches (locally) in two places. Between those places, the distance to the pool wall is less than the radius of the expanding ripple, and so is closer than the places where the ripple intersects the wall.



Problem 10 (Student page 31)

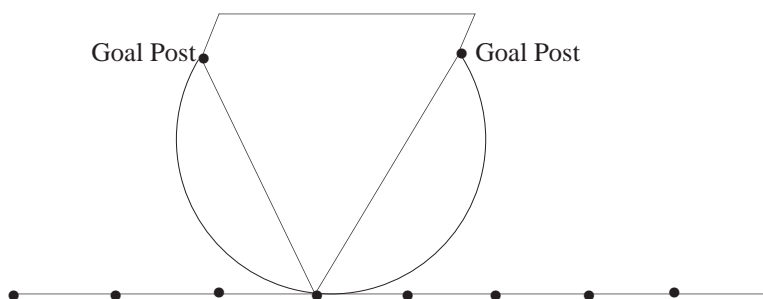
Given any line and any two points not on the line, can you construct (using a straightedge and compass or geometry software) a circle that passes through the two points and is tangent to the line? That's an interesting challenge.

- a. Draw the perpendicular bisector of the line segment connecting the goal posts. Call the intersection of this line with the line of players P . Draw the circle whose center is M , the midpoint of the segment connecting the goal posts, and whose radius is MP . This circle will be tangent to the line of players at P . Now use Problem 21 of Investigation 6.2.
- b. The angle first increases and then decreases as you move from left to right across the field. The graph on the next page (created with Cabri Geometry II™) gives a more quantitative picture of how the angle varies. (To create such a graph, set the player on a horizontal line, measure the angle to the two goal posts in some suitable unit, and then transfer that numerical measurement, as if it were a length, to a vertical ray from the player. Trace the resulting point as the player moves.) The angle is at its greatest when the player is centered between the goal posts (given that the line is parallel in this case); as the player moves toward either side of the field, beyond either goal post, the angle decreases most rapidly.



- c.** In theory, for any less-than-optimal angle, there are exactly two spots along the line of a player lineup that afford that angle. They are symmetrically placed on either side of the optimal spot. In practice, in this case, there would be a discrete number of players standing at particular spots along the line. You'd use the theory to inform practice.
- d.** No matter how small the angle, there are spots from which the “view” of the goal posts fills only that angle. For an angle of 1° , those spots are definitely not on the “line” (segment) the coach drew on the playing field, but will exist in two places on the infinitely long line *through* the coach's line. The many points that afford a kicking angle of 1° lie on a circle whose center has a kicking angle of 2° . This is a *very* large circle; the only parts of the circle that would actually lie on the playing field would be, in a player's words, “practically in line with the goal posts.”

Problem 11 (Student page 31) The sketch should be a circular arc through the goal posts.



The relevant theorem is developed in Problems 19–21 in Investigation 6.2:

THEOREM *Inscribed Angles*

Angles inscribed in the same circular arc have the same measure.

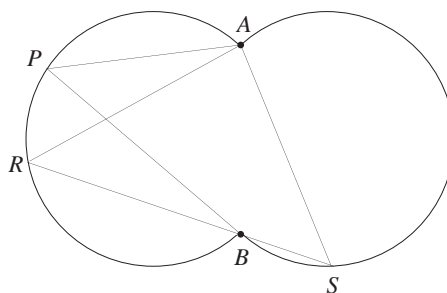
This problem asks for a weak converse, also true, of this *Inscribed Angles* theorem:

THEOREM *Inscribed Angles Converse*

The locus of vertices of congruent angles whose rays pass through two fixed points is two (congruent) arcs joined at their endpoints.

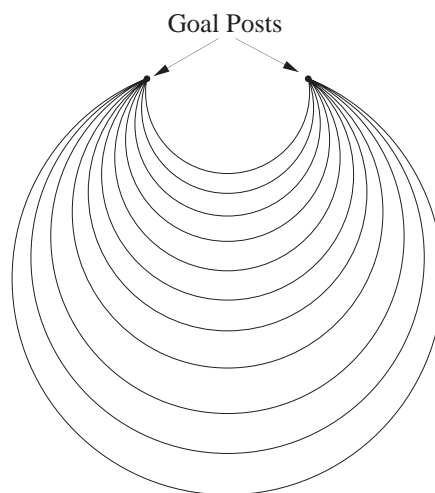
To be a true converse, we'd expect the theorem to end: "... an arc." But look at the picture:

$\angle APB$ is congruent not only to $\angle ARB$, but also to $\angle ASB$.

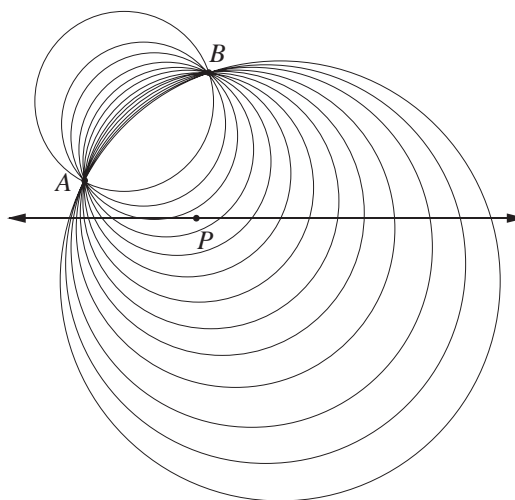


For practical reasons (soccer is played only in front of the goal posts) the second arc is of no interest here, so the players stand on one circular arc. They would all have a kicking angle of 40° .

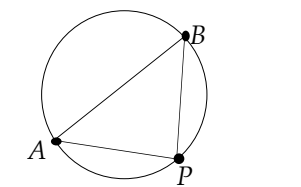
Problems 12–13 (Student page 32) The contour lines are all circular arcs because the angles inscribed in each circle that passes through the goal posts have the same measure, whereas the angles along a line in front of the goal will either increase or decrease. As you move away from the goal, the numbers decrease. Larger arcs are contour lines for smaller kicking angles; an additional contour line will represent the value for a kicking angle between the values of the lines between which it was placed.



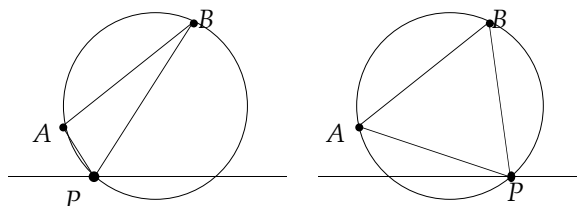
Problem 14 (Student page 32) There are some very important ideas here. A picture of the full contour map and running path looks like this:



Some of the contour lines do not cross your path. They afford better kicking angles, but are inaccessible (if you remain on the path).



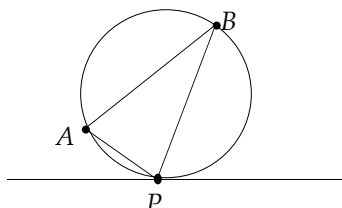
Intuition has already told us that for any less-than-optimal kicking angle, there are *two* places along your path that give that angle. Both of the following pictures show $\angle APB$ at 12° below its maximum. The contour circle intersects the path at the two different places where P causes $\angle APB$ to have this value.



This notion of tangency of contour lines optimizing some value is revisited often in mathematics. The horizontal tangent to the curve that students study in calculus is one example.

If we cross contour lines while running along the path, the value of the angle is changing, so we cannot be at its maximum value. The contour line of the maximum value is one that we just touch, but do not cross (for, if we cross it, we are at a higher value).

When $\angle APB$ is at its maximum, its contour line is tangent to the path.



Problem 17 (Student page 33) As you drag P around, the value for the function f will change. But there are certain sets of points for which the function has some

particular constant value. These sets of points, or paths, of constant total distance ($PA + PB$) form ellipses, with A and B as foci. One way to visualize this is to imagine (or build a model) of P as a washer on a string attached at either end to A and B .

Problem 18 (Student page 34) Here each contour line is a circle. Contour lines are lines of fixed value for the function. Function g evaluates distance from a single point, so curves of constant value for g are located at constant distances from a fixed point, which means they are circles. The full contour map is a set of concentric circles around C .

Problem 19 (Student page 34) Points in space a constant distance r from a line ℓ lie on a cylinder of radius r , with ℓ as its axis. (Like ℓ , the cylinder will have infinite length.) If in looking for these points you restrict your search to a plane that contains ℓ , you will find the points lie on the intersection between the cylinder and that plane: two lines parallel to ℓ . This example in the plane is the first in this module in which contour lines are not single, connected curves. That is, the contour “line” for “one inch away” (when the value of f is 1 inch) is really two parallel lines, one on either side of ℓ .

Problem 20 (Student page 35)

- a.–c.** The contour lines here are ellipses. The numbers that could belong to the “new” contour line would have to be in between the values for the original second and third contour lines.
- d.–e.** The way to use the contour lines to locate the point along some arbitrary line or circle (or even arbitrary curve) for which f produces the smallest value follows exactly the same logic as described in the solution for Problem 14. As long as your arbitrary line (or curve) does not intersect \overline{AB} , there will be an ellipse to which it is tangent. That point will yield the smallest value for f . (You may not have *drawn* that particular ellipse, and, if not, it will lie inside the smallest ellipse your line intersects.)
- f.** Function f computes the sum of the distances from a point to A and B . The Triangle Inequality tells us that that sum cannot be less than the distance between A and B . Thus, values smaller than that are not in the *range* (cannot be values) of f —no contour lines would correspond to them.

The *axis* is the line through the center of a cylinder, perpendicular to its faces.

This problem foreshadows Problems 27, 36, and 41.

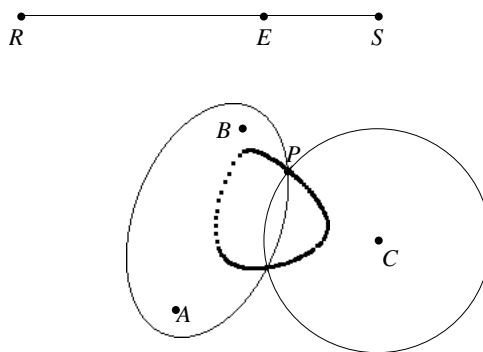
If your line intersects \overline{AB} , then the point of intersection yields the smallest value for f .

The string-and-nails technique suggested in the Student Module encourages you to think of physical models and specifically foreshadows models to be encountered later.

Problems 21–22 (Student page 35) These two problems are extensions of the thinking that is being developed here. For both problems the contour plots might be sketched by hand or using the “slider point” method with geometry software. For a detailed description of the latter, see the section “Drawing Curves with Geometry Software” on pages 24–32 in the *Teaching Notes* for Investigation 6.4.

Here is a brief outline of the method using Problem 21 as the example. Construct a segment, say \overline{GJ} of length k , with a slider point, H , that breaks the segment into two parts. Next, construct two circles of radius GH and $HI = \frac{1}{2}HJ$ centered at E and F . As H is slid back and forth along \overline{GJ} the intersections of the two circles, Z and Z' , will trace out the egg-shaped set of points that represent the constant sum $k = ZE + 2ZF$.

For Problem 22, the method is analogous. The slider point breaks a segment into two parts, the sum of whose lengths is constant. These parts can be used to generate figures other than circles; the set of possible loci that students can build depends only on the available primitives. To solve Problem 22, the authors had to wait until *Cabri Geometry II*TM built in conic sections as primitives. What we want here is “a generalized ellipse with three foci,” that is, the locus of points P so that $PA + PB + PC$ is constant. It can be obtained from the intersection of an ellipse and a circle:



$$PA + PB + PC = RS$$

To generate the locus, slide E back and forth along \overline{RS} .

In this sketch, \overline{RS} is divided at E . An ellipse with major axis ES is constructed with foci A and B , and a circle with radius RE is constructed with center C . P is the intersection of the ellipse and the circle.

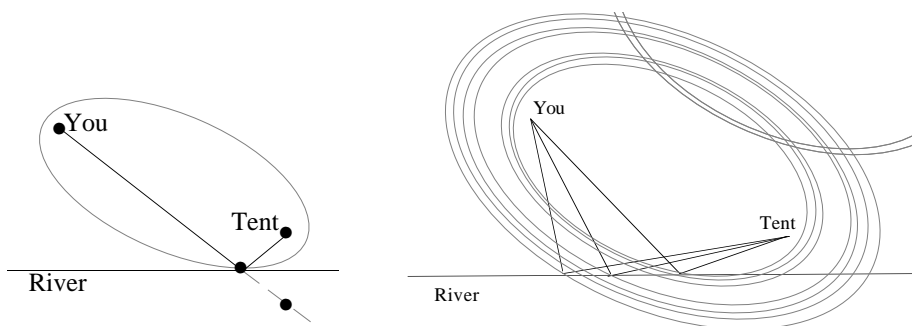
Problem 23 (Student page 36) The contour lines are circular arcs like those in Problems 11–14. An important but subtle point left unsaid there is that because $h(A)$ and $h(B)$ are not defined points, A and B cannot be included on the contour plot.

($\angle APB$ neither has a natural definition when P coincides with either A or B , nor can a limiting argument be used to assign an unambiguous meaning to it.)

Problem 24 (Student page 36) If P is above \overleftrightarrow{AB} and $m\angle APB = 90^\circ$, this contour line is a semicircle. $m\angle APB$ is less than 90° when P is outside this contour line and greater than 90° when P is inside the contour line.

Problems 27–29 (Student pages 37–38) We use ellipses again. The loop-around-two-nails technique described in Problem 31 helps in drawing these by hand. The information immediately preceding Problem 32 will help in constructing them with software.

In the figure on the left below, the ellipse shown is the contour line for the optimal spot. All the points on the ellipse have the same value as the best spot: the minimum total distance from your spot (You) to the river to the Tent. Other spots on the river represent greater values for the function that calculates the total distance. The contour lines for those values will be larger ellipses, and will not be tangent to the line of the river but will intersect it.



The smaller the total distance k is (computed as $AP + BP$), the “flatter” the ellipse. The larger k is, the more the curve resembles a circle. A way to think about this is to imagine extreme cases. Very large values of k dwarf the distance between the two foci. Thus the curve is (more or less) the set of all points k distance from that *one* place: a circle. Very small values of k (it cannot be smaller than the distance between the two foci) generate a line segment between the two foci.

Problem 30 (Student page 39) Yes, the two definitions are equivalent. Suppose points A and B are fixed. The first definition says that if you pick a number k , the set of all points P for which $PA + PB = k$ is an ellipse. This is exactly the same set

of points as in the contour line consisting of all points P such that $f(P) = k$, which is the second definition. For any positive number k , both definitions define the same locus of points, so they are equivalent.

Problem 31 (*Student page 39*) The important point is that the string is kept taut. If the tip of the pencil is at point P , then the sum of the distances from P to the nails is equal to the length of the string. This length doesn't vary as P moves, so all the points traced by the pencil will have the same total distance to the two nails.

If A and B are the locations of the nails, what you're really doing is constructing a contour line for the function $f(P) = PA + PB$ from the previous problem. If the length of the string is k , you are drawing all the points P for which $f(P) = k$.

Problems 32–34 (*Student page 41*) If k is too small, the ellipse does not intersect the riverbank. You cannot get to the river and back to the Tent with such little running!

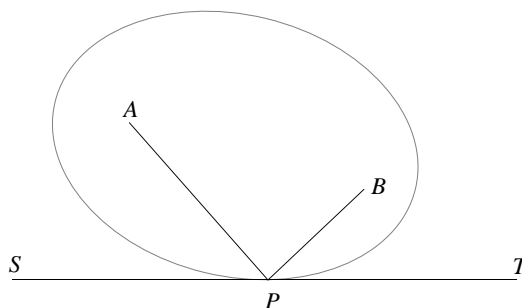
If k is too large, the ellipse for that value of k intersects the riverbank twice. Inside the contour line, between these two intersections, is some spot or a number of spots, along the riverbank that will yield a shorter distance. Those better spots have contour lines of their own. If any of these new contour lines intersect twice, then the places between those intersections are inside the curve and are therefore even better places to be.

Only one contour line is tangent, and only for the value of k of that contour line is there no better place—no place inside the curve—to which You could run. Thus the contour line method should agree with the reflection method discussed earlier.

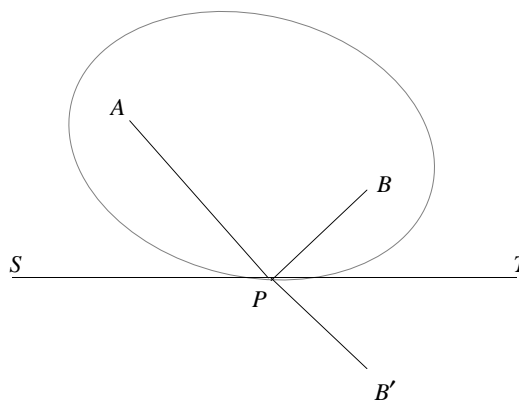
Problem 35 (*Student page 41*) Place a ball on each focus of an elliptical billiard table. Shoot one ball in *any* direction hard enough to travel a distance equal to the major diameter of the table, and it hits the other ball! Why?

Here's a way to remember the definition of an ellipse: "Points a constant total distance k from A and B lie along an ellipse."

Draw an ellipse with foci A and B , and draw a tangent \overleftrightarrow{ST} to the ellipse, intersecting it at P .



We have seen that, if you are at A and your burning tent is at B , then the best place to stop at the river (\overleftrightarrow{ST}) is at P . But the best place to stop at the river is also at the place where the line between A and the reflection of B over \overleftrightarrow{ST} intersects \overleftrightarrow{ST} :

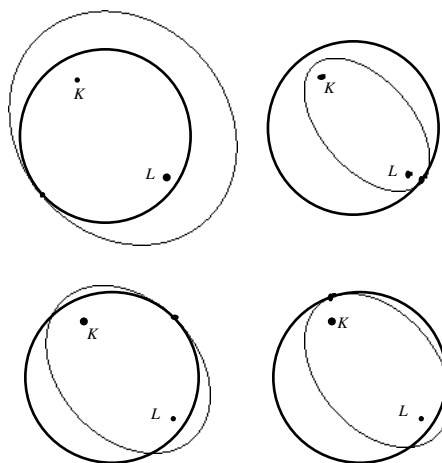


B' is the image of B over \overleftrightarrow{ST} .

So, P must be the intersection of \overleftrightarrow{ST} with $\overleftrightarrow{AB'}$. But since $\angle SPA \cong \angle TPB'$ (vertical angles) and $\angle TPB' \cong \angle TPB$ (one is the reflection of the other), we have $\angle SPA \cong \angle TPB$.

This problem is essentially equivalent to Problem 20 of this investigation.

Problem 36 (Student page 42) The set of points that represent a constant sum of distances to K and L will be ellipses, and the spot where the one ellipse is tangent to the circular edge of the pool is the spot that minimizes the total length Natasha has to swim. The problem is that the ellipses (the contour lines) can be tangent to the circle in several different ways:



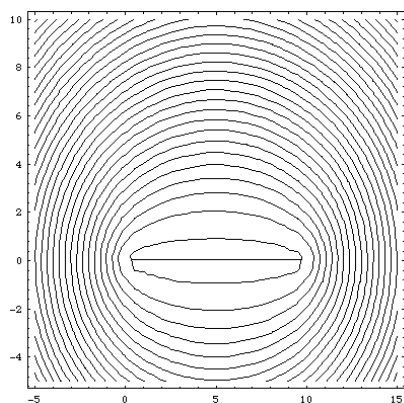
Which of these ellipses is tangent in the best spot for Natasha to leave her sunglasses? What properties do the other intersections have?

Problem 37 (Student page 42)

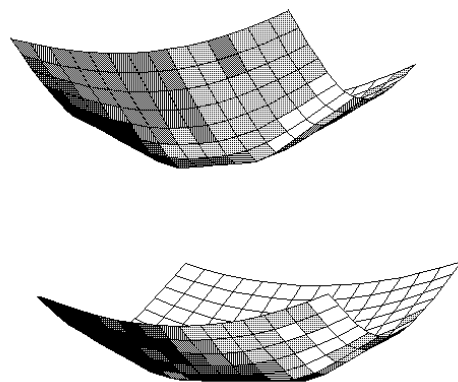
- a. There are two main distinguishing differences. A topographic map is a two-dimensional picture that draws attention to features of constant value, whereas a relief map, or surface plot, is a three-dimensional model or map. (However, people often refer to a *picture* of a relief map as a relief map—that can be confusing.) The second important difference is in the nature of the representation of features. Topographic maps and contour plots depict information in discrete steps; there are gaps between values, and many of the contour lines cannot be pictured. Relief maps and pictures of them are continuous (there are no gaps or holes in between any of the points or features represented).
- b. The relief map would look like an upright cone, balanced on its point. The topographic map would look like a bullseye—a bunch of concentric circles.
- c. The relief map for g is an elliptical cone opening upward or a bowl shape with a line segment for its foundation instead of a point. The line segment lies

Mathematica® is a piece of software designed specifically for doing mathematics. It will do algebraic manipulations and make pictures like these, among other things.

on a base plane; imagine vertical rays perpendicular to the plane from each endpoint. Successive planes will cut ellipses with foci at the intersections with the vertical rays. Here are some pictures generated in *Mathematica*.

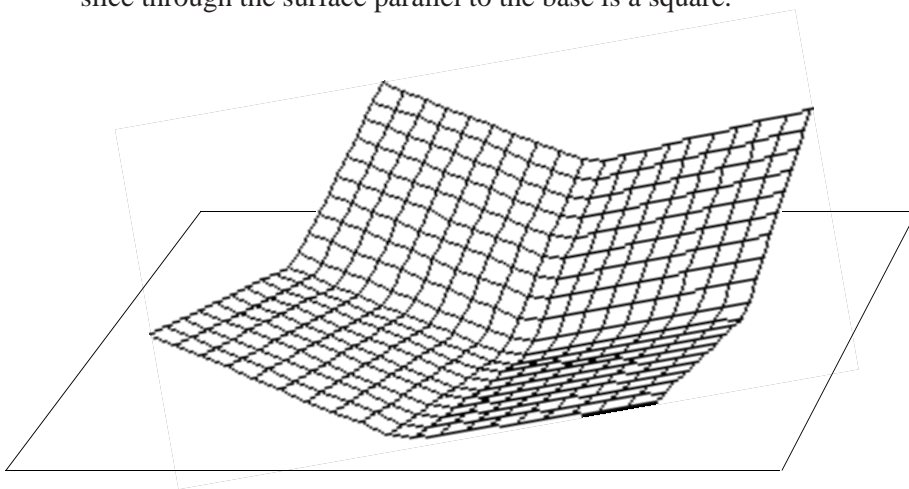


The contour plot



Two views of the surface plot

- d. The contour plot would look like a series of equally-spaced concentric squares. Plotting packages that create pictures like this compound the difficulty of visualizing the contour plot by cutting off the surface in ways that do not necessarily fit the contour of the surface. That is why we see irrelevant corners on this surface. Trimmed differently, it would be easier to see that this surface is the nonbase surface of an inverted square pyramid; thus, each slice through the surface parallel to the base is a square.



Problem 38 (*Student page 43*) First you may want to decide how much longer it takes you to run with a full bucket than with an empty bucket. If it takes you twice as long, then the problem is similar to Problem 21 (except that here we are looking only at points along a line, not any point in the plane). The apparatus shown in the sidenote to Problem 21 on page 35 of the Student Module will help in drawing the contour lines. See commentary “Using Technology” of Investigation 6.2 in the section of the *Teaching Notes* for Problems 3 and 8 of Investigation 6.2. Coordinate techniques can also be used to generate a function whose graph shows the optimal spot. (See, for example, page 9 in the “Mathematics Connections” section of the *Teaching Notes* for Investigation 6.2.)

Problem 40 (*Student page 44*) If you think of the best viewing angle as the widest angle, then this is just another version of the soccer problem (see Problem 8 of Investigation 6.3). Suppose your location on the highway is X , and A and B are the two front corners of the building. Then you want to find which position for X maximizes the measure of angle $\angle AXB$. As in the soccer problem, the solution is to construct a circle through points A , X , and B , and then find the location for X that makes the circle tangent to the line representing the highway.

Problem 41 (*Student page 44*) This problem is essentially equivalent to Problems 20 and 36. See the solutions for those problems.

Problem 42 (*Student page 45*) See the solution for Problem 38. This is the same problem except that L has been reflected over the shoreline.

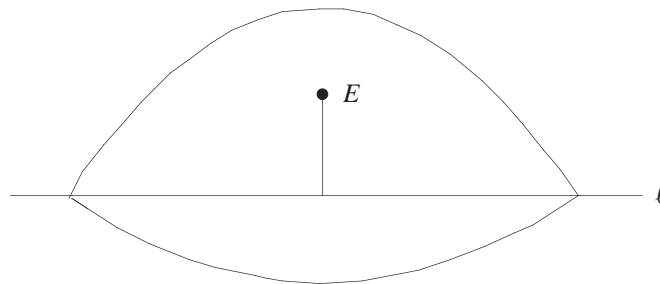
Problem 43 (*Student page 45*) There are at least two problems here. The proposed solution is to find the smallest ellipse with A and B as foci that does not cross the refuge. But there is no need for the entire ellipse to exist outside of the refuge. In other words, there might exist an ellipse with the property that part of it intersects the refuge, but part of it doesn't; it is on this second part where a good location for the recreation center might lie. Restricting attention to only those ellipses which completely contain the refuge might rule out some solutions.

Second, the given answer doesn't consider some other important questions. For example, the recreation center should be conveniently located to both cities. It would not be fair if the people in one city had to drive much longer than the people in the other city, even if such a solution minimized the total amount of new roadway needed.

Problem 44 (*Student page 45*) In the figure on the next page, two contour lines are shown. The one smallest in value is the perpendicular line segment from point E to ℓ . Any point along that line segment will result in the same value for function f .

Can you generate these contour lines with your geometry software?

The second contour line is the union of two curves, and each of these curves is a piece of a parabola. The best way to see this is to use analytic geometry. Set things up so that ℓ is the x -axis and E is on the positive y -axis.



Problem 45 (*Student page 46*) The function describes concentric spheres. A contour plot, because it's two-dimensional, would look like a series of concentric circles.

USING OPTIMIZATION

Problem 2 (Student page 47) If a mathematical statement has been proved, it is known as a theorem. If the statement is something that is believed to be true, such as an educated guess, it's a conjecture. There exist a number of famous conjectures in mathematics—unproved statements that have existed for years and which most mathematicians believe to be true. These are still conjectures, however; no matter how many people believe them, they remain conjectures until they are proved.

Problem 3 (Student page 47)

- a. The measure of $\angle NLP$ is 90° , since \overline{NL} is perpendicular to line \overleftrightarrow{LM} .
- b. When we talk about distance from Q to \overleftrightarrow{LM} , we mean the length of the shortest path from Q to \overleftrightarrow{LM} . The shortest path from a point to a line is the perpendicular line segment between them; this would be \overline{QM} . Hence, the distance is 2.2 cm.
- c. We are not given the length of \overline{LP} , but we do know the lengths LN and PN , and we know that we have a right triangle with vertices L , N , and P . By the Pythagorean Theorem,

$$\begin{aligned}(LP)^2 + (LN)^2 &= (PN)^2 \\(LP)^2 &= (PN)^2 - (LN)^2 \\(LP)^2 &= (5.2)^2 - (4.1)^2 = 10.23.\end{aligned}$$

So $LP = \sqrt{10.23} \text{ cm} \approx 3.2 \text{ cm}$.

- d. $PN = 5.2 \text{ cm}$ and $PQ = 8.2 \text{ cm}$. Thus, P is not equidistant from N and Q ; P is closer to N .
- e. Yes, \overline{NL} and \overline{QM} are parallel because they both are perpendicular to \overleftrightarrow{LM} .
- f. As P moves to the right along \overline{LM} , the length NP will increase. As P moves to the right, $\triangle NLP$ changes; the height \overline{NL} stays fixed, but the base \overline{LP} becomes longer. This means that the hypotenuse \overline{NP} lengthens also; thus, NP increases.

Here's a way to be certain that the hypotenuse gets longer as P moves to the right along \overline{LM} : Suppose P moves to the right to a new point P' . Look at the right triangles $\triangle NLP$ and $\triangle NLP'$. The Pythagorean Theorem says that

$$(NL)^2 + (LP)^2 = (NP)^2$$

and

$$(NL)^2 + (LP')^2 = (NP')^2.$$

It is clear that $LP' > LP$, so that $(LP')^2 > (LP)^2$. Therefore, the equations above tell us that $(NP')^2 > (NP)^2$, so $NP' > NP$ (as they are both positive quantities).

Problem 4 (Student page 48)

- a. Imagine that \overline{NQ} is drawn and look at $\triangle NPQ$. By the Triangle Inequality, $NP + PQ > NQ$, so the given statement is false.
- b. The given statement is false. The length $NP + PQ$ will vary as P moves. In fact, you know where P should be to minimize $NP + PQ$; this is the burning tent problem.
- c. The given statement is true. This is the definition of midpoint.
- d. The given statement is false. If P is at the midpoint of \overline{LM} , then $LP = PM$; call this length x (or calculate x using the given measurements). Now apply the Pythagorean Theorem to $\triangle LNP$ and $\triangle MPQ$:

$$(4.1)^2 + x^2 = (NP)^2$$

$$(2.2)^2 + x^2 = (PQ)^2.$$

If $NP = PQ$, then the equations above would show that $4.1 = 2.2$! Therefore, the two lengths are not equal.

- e. By the Triangle Inequality, $NL + LP > NP$, so the given statement is false.
- f. False; if $\angle NPQ = 90^\circ$, then we know that

$$m\angle LPN + m\angle QPM = 90^\circ$$

since a straight line measures 180° . We also know that

$$m\angle LPN + m\angle LNP = 90^\circ$$

and

$$m\angle QPM + m\angle PQM = 90^\circ$$

since the sum of the angle measures in a triangle is 180° .

The first two equations above tell us that

$$m\angle QPM = m\angle LNP,$$

while the first and the third tell us that

$$m\angle LPN = m\angle PQM.$$

Therefore, if $NP = PQ$, $\triangle LNP$ and $\triangle PQM$ would be congruent by the ASA Postulate. That would mean that

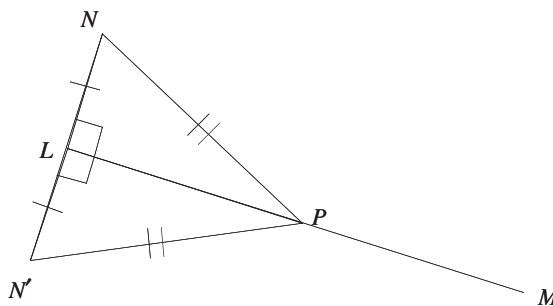
$$LP = QM = 2.2$$

and

$$LN = PM = 4.1,$$

implying that $LM = 2.2 + 4.1 = 6.3$. However, using the measurements we were originally given (see Problem 3c), we can calculate that $LM = \sqrt{10.23} + \sqrt{62.4}$, which is approximately 11, not 6.3. Although P is allowed to move, \overline{LM} is fixed, so its length can't vary. Thus, the given statement is false.

- g. By reflecting N about \overline{LM} , we get the point N' which forms a line segment $\overline{N'L}$ perpendicular to \overline{LM} , and also has the property that $N'L = NL$. See the picture below. Thus, \overline{LM} is the perpendicular bisector of $\overline{NN'}$. Recall that any point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment. This means that $PN = PN'$. Clearly $LP = LP$, so by the SSS Postulate, $\triangle NLP \cong \triangle N'LP$. This tells us that $m\angle NPL = m\angle N'PL$.



Problem 5 (Student page 48) If you picture the bottom of the pool as a straight line, l , the problem becomes the same as the burning tent problem. First of all, the swimmer tries to make a long dive and enter the water as close to the opposite end of the pool as possible. Call the point where the swimmer enters the water A . Call the point which is at the top of the water at the far end of the pool B . Then the problem is just to find the point P on line l which minimizes the length $AP + PB$, the burning tent problem.

Problem 6 (Student page 48) Recall the solution to the burning tent problem: we reflect A about the line l (the river) to form a new point A' ; we construct $\overline{A'B}$, and then

find point P where $\overline{A'B}$ intersects l . Call this point the “best spot” for P . We want to show that as P varies, the length of the path $\overline{AP} + \overline{PB}$ will also vary, and that this length is shortest when P is located at the best spot.

Because P is on the perpendicular bisector of AA' , we know that $PA = PA'$, so $AP + PB = A'P + PB$. Therefore, it suffices to look at the path $\overline{A'P} + \overline{PB}$. As P varies, so does this path. At exactly one spot (the best spot) $A'P + PB$ will be a straight line; at all other spots it will *not* be straight. But $\overline{A'P} + \overline{PB}$ is a path from A' to B , and we know that the shortest path between two points along a straight line. Thus, when P is at any place other than the best spot, $A'P + PB$ will be longer than it is when P is at the best spot. Therefore, $AP + PB$ is also minimized when P is at the best spot; the length $A'P + PB$ does not stay constant.

Problem 7 (Student page 49) X will not always be the midpoint. It makes sense to try some special cases, such as right triangles, isosceles triangles, and obtuse triangles. Point X will be the midpoint of \overline{AC} if and only if $\triangle ABC$ is isosceles. Using what you know from the burning tent problem, see if you can write a proof of this.

Consider $\triangle ABC$. Let $\angle 1 = \angle AXM_1$ and let $\angle 2 = \angle CXM_2$. Recall from the burning tent problem that because X is the point that makes path $M_1X + XM_2$ as short as possible, we know that $m\angle 1 = m\angle 2$.

First, assume that X is the midpoint of \overline{AC} ; thus $AX = XC$. Since X is a midpoint, segments $\overline{XM_1}$ and $\overline{XM_2}$ are *midlines* of $\triangle ABC$. (A *midline* of a triangle connects the midpoints of two consecutive sides.) We then apply the following theorem:

THEOREM

A midline of a triangle is parallel to the third side of the triangle.

This is the half of the Midline Theorem, which has been discussed in earlier modules and is proved in the module *The Cutting Edge*.

This theorem tells us that segments $\overline{XM_1}$ and \overline{BC} are parallel, as are segments $\overline{XM_2}$ and \overline{AB} . This implies that

$$m\angle 1 = m\angle M_2CX,$$

and

$$m\angle 2 = m\angle M_1AX,$$

respectively. Therefore, since $m\angle 1 = m\angle 2$, we have that

$$m\angle M_2CX = m\angle M_1AX.$$

This lets us conclude that $\triangle ABC$ is isosceles.

For the other half of the proof we assume that $\triangle ABC$ is isosceles, and show that X is the midpoint of \overline{AC} . Since the triangle is isosceles, we know that

$$m\angle M_1AX = m\angle M_2CX$$

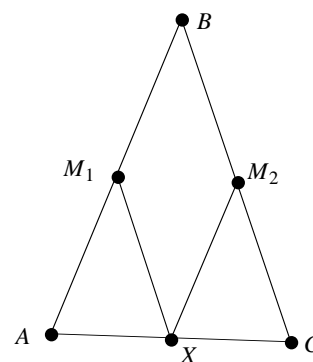
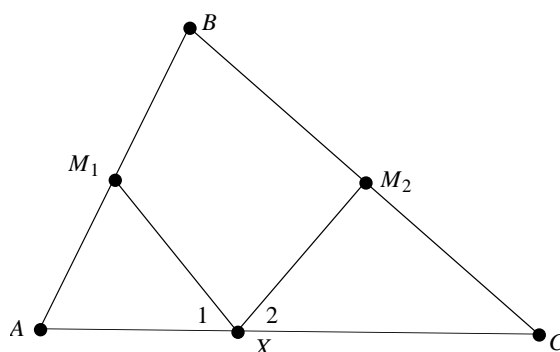
and also that

$$AM_1 = CM_2.$$

Since $m\angle 1 = m\angle 2$, we see that $\triangle AM_1X \cong \triangle XM_2C$. Therefore,

$$AX = XC,$$

so X is the midpoint of \overline{AC} .



The shortest distance exists when $m\angle AXM_1 = m\angle CXM_2$.
 $\triangle ABC$ on the right is isosceles.

Problem 8 (Student page 49)

- a. Point T will be inside the new square. This might be hard to see intuitively, but you can reconstruct the situation on the computer or with paper and ruler to try to discover the correct answer. A proof of this is given below the answers to the questions for part b.
- b. The area of the first farm is 4 km^2 . Each child's farm has a perimeter of 6 km and area of $\sqrt{3} \text{ km}^2$. (Divide an equilateral triangle into two 30–60–90 triangles to find this area.) The total perimeter of the land owned by the family is 16 km, while the total area of this land is $4 + 4\sqrt{3} (\approx 10.9) \text{ km}^2$. The farm with maximum area given a perimeter restriction is circular. To

find the area of a circle with circumference 16 km, first find the radius:

$$C = 2\pi r$$

$$r = \frac{C}{2\pi} = \frac{16}{2\pi} = \frac{8}{\pi},$$

So the radius of the circle is $\frac{8}{\pi}$ cm. Now find the area:

$$A = \pi r^2$$

$$= \pi \left(\frac{8}{\pi} \right)^2$$

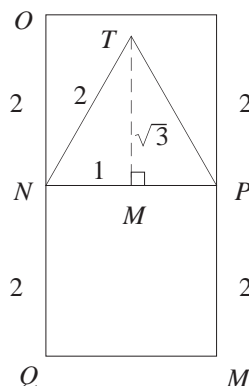
$$= \frac{64}{\pi} \approx 20.4.$$

The maximum possible land area is 20.4 cm^2 .

Iso- is a prefix meaning *equal* or *uniform*. What do you think *isoperimetric* means?

It is a fact that a circle maximizes area when the perimeter is fixed. Indeed, this is the *isoperimetric* problem. The last section of this module, “The Most Area,” is devoted to this problem. The solution can be conjectured at this point and will be proved later.

Proof that T is inside the square: Let M be the midpoint of \overline{NP} . Since $\triangle NTP$ is equilateral, \overline{TM} is an altitude and $TM = \sqrt{3}$. Since $\sqrt{3} < 2$, $TM < ON$, so T is inside the square.



Problem 9 (Student page 50) The best solution here is for the farmer to once again purchase a circular region of land surrounding her farm. Suppose the original farm is a square with area of $x^2 \text{ km}^2$. Then each child’s farm must have an area of $\frac{1}{4}x^2 \text{ km}^2$, so the total area of land will be $2x^2 \text{ km}^2$.

Since the amount of fencing needed for the original farm will never change, the total amount of fencing needed will be minimal when the outside perimeter of the family's land is minimal. Therefore, we have a fixed area, and want to find the shape enclosing this area with the smallest perimeter. The answer is once again a circle; this too will be dealt with in the last section of this module, "The Most Area" (see Problem 8 of Investigation 6.20). The best idea is to purchase a circular area surrounding the original farm and then divide the area outside of the original farm into 4 equal pieces for the children. Since we know that the area of the circle is $2x^2 \text{ km}^2$, we can find the radius and then use the radius to find the circumference.

$$\begin{aligned} A &= \pi r^2 \\ 2x^2 &= \pi r^2 \\ r^2 &= \frac{2x^2}{\pi} \\ r &= x\sqrt{\frac{2}{\pi}} \text{ km,} \end{aligned}$$

and

$$\begin{aligned} C &= 2\pi r \\ &= 2\pi x\sqrt{\frac{2}{\pi}} \\ &= (2\pi x) \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{\pi}} \\ &= \frac{2\pi x\sqrt{2\pi}}{\pi} \\ &= 2x\sqrt{2\pi} \text{ km.} \end{aligned}$$

Thus, the total amount of fencing needed will be $2x\sqrt{2\pi} \text{ km}$.

Problem 10 (Student page 50) The volume of the typical box is 273 cm^3 ($l \times w \times h$). The size and shape of a rectangular-based juice box that will use the least cardboard for its surface area will always be a cube. Because in this problem you were given that the fixed fluid volume of approximately 250 ml is contained by a shape the volume of which is 273 cm^3 , the question is reduced to finding the dimensions of a cube of that volume (a cube of edglength $\sqrt[3]{273}$).

The cube is the rectangular box that minimizes surface area. Consider the problem in two dimensions. Instead of a rectangular box we have a rectangle, instead of volume we have area, and instead of surface area we have perimeter. You already know that of all rectangles with a fixed area the square has the smallest perimeter (see Problem 24 of

Investigation 6.3 of the Student Module). If the answer is a square in two dimensions, this kind of reasoning by analogy leads us to believe it's a cube in three dimensions!

You could also try calculating several examples of different boxes with that same volume of 273 cm^3 . For instance, if the box has dimensions $180 \text{ cm} \times 1 \text{ cm} \times 1 \text{ cm}$, the surface area equals 722 cm^2 , which is quite large. If the dimensions are $18 \text{ cm} \times 5 \text{ cm} \times 2 \text{ cm}$, the surface area is 272 cm^2 . In the $10 \text{ cm} \times 6 \text{ cm} \times 3 \text{ cm}$ case, the surface area is 216 cm^2 , and with dimensions $5 \times 6 \times 6$, the surface area equals 192 cm^2 . It seems as if the surface area becomes smaller as the sides of the box become closer to each other in length. The pattern in these results would also lead one to surmise that perhaps a cube is the best solution for minimal surface area.

To prove that the answer is, in fact, a cube, we make use of the *arithmetic-geometric mean inequality* for three variables. It states that if a , b , and c are positive real numbers, then

$$\sqrt[3]{abc} \leq \frac{a + b + c}{3}.$$

See Problem 5 of
Investigation 6.3 for details.

Recall that we used this earlier for two variables: if a and b are positive real numbers, then

$$\sqrt{ab} \leq \frac{a + b}{2}.$$

The Proof If x , y , and z represent the sides of the box, the surface area is given by $2(xy + xz + yz)$. To minimize surface area, it suffices to minimize the quantity $xy + xz + yz$. We will call this quantity S , so $S = xy + xz + yz$. Let V represent the volume.

Remember that $V = xyz$. Now let

$$a = xy$$

$$b = xz$$

$$c = yz.$$

The inequality says that

$$\sqrt[3]{abc} \leq \frac{a + b + c}{3},$$

So with substitution, we have

$$\begin{aligned}\sqrt[3]{(xy)(xz)(yz)} &\leq \frac{xy + xz + yz}{3} \\ \sqrt[3]{(xyz)^2} &\leq \frac{xy + xz + yz}{3} \\ \sqrt[3]{(V)^2} &\leq \frac{S}{3} \\ 3\sqrt[3]{(V)^2} &\leq S.\end{aligned}$$

Therefore, S must be minimal when

$$S = 3\sqrt[3]{(V)^2}.$$

This happens when $x = y = z$ because when this is true we have $x = \sqrt[3]{V}$, and $x^2 = \sqrt[3]{(V)^2}$, so it follows that

$$S = x^2 + x^2 + x^2 = 3\sqrt[3]{(V)^2}.$$

For a proof of the arithmetic-geometric mean inequality, see the “Mathematics Connections” section of the *Teaching Notes*.

A beautiful attempt at the “real essence of mathematics” along a path from “the very elements to ... the substance of modern mathematics” is found in *What Is Mathematics?* by Richard Courant, Herbert Robbins and Ian Stewart (Oxford University Press, New York, 1996).

Surface area is minimized when $x = y = z$, that is, when the box is a cube.

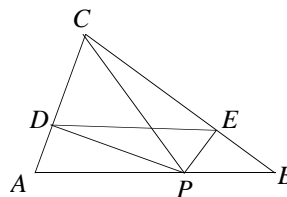
Problem 11 (Student page 50) The point of tangency is the point on the tangent line closest to the center of the circle. A segment drawn from the center to any other point on the line must at least contain a radius of the circle and therefore is longer than the radius. A radius drawn to the point of contact with the tangent line is perpendicular to the tangent.

Problem 12 (Student page 51) This is a classic problem, due to Hermann Amandus Schwarz. Schwarz (1843–1921) was a distinguished mathematician at the University of Berlin and one of the great contributors to modern function theory and analysis. The solution is quite surprising: The smallest perimeter triangle is obtained by connecting the feet of the altitudes of the outside triangle. For a proof, see Courant, Robbins, and Stewart; and for a connection with an unsolved problem in geometry, see Klee and Wagon’s book. (See the first sidenote on the next page.)

In *Unsolved Problems in Plane Geometry and Number Theory* by Victor Klee and Stan Wagon (Mathematical Association of America, 1991), the authors present some of the background for a number of unsolved problems, as well as some of the techniques used to produce partial results.

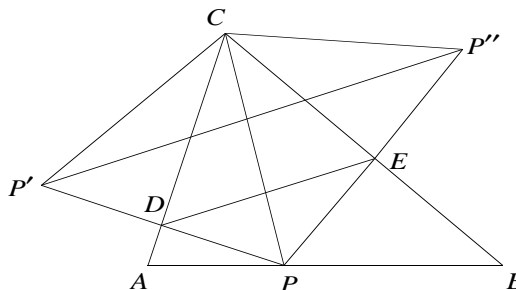
Geometry software is especially convenient here in coming up with a conjecture. The proof, however, is tricky.

Problem 13 (Student page 51) There are many ways to approach this problem. Variations on the solution for P that follows show the interplay between deduction and experiment, as well as the use of habits of mind such as looking at special cases and performing thought experiments. A little experimenting shows that the best spot to minimize DE (subject to the constraints that $\overline{PD} \perp \overline{AC}$, and $\overline{PE} \perp \overline{BC}$) seems to be at the foot of the altitude from C to \overline{AB} . Why?



Let's work backwards: The claim is that DE is a minimum when $\overline{CP} \perp \overline{AB}$. That is, we think that DE is a minimum when CP is a minimum. That would follow if $DE = CP$, which would be true if $\angle C$ were a right angle; then $DCEP$ would be a rectangle and the diagonals of a rectangle are congruent. So we have a proof in the special case where $\angle C$ is a right angle.

The proof that makes use of $DCEP$ when it's a rectangle doesn't generalize easily, but it *does* generalize. It turns out that, while $DE \neq CP$ in general, DE depends on CP in such a way that one is a minimum precisely when the other is. Here's how to see this. First, reflect P over \overline{AC} and \overline{BC} to obtain P' and P'' ; then connect up some points:



Now convince yourself of three facts:

Fact 1. $P'P'' = 2DE$ (by the Midline Theorem);

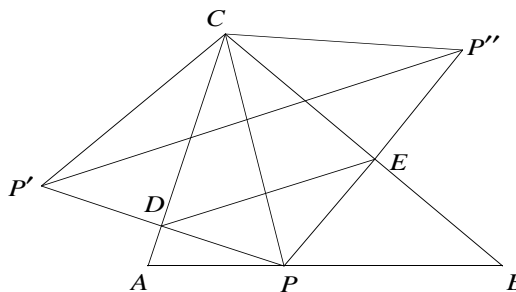
Fact 2. $CP' = CP = CP''$ (they are all reflection images of each other);

Fact 3. $m\angle P'CP'' = 2m\angle ACB$ ($\angle PCA \cong \angle P'CA$ and $\angle PCB \cong \angle P''CB$).

Two isosceles triangles that have the congruent vertex angles are similar by SAS for similarity. Try this with your geometry software. Do the triangles look similar?

So, by Fact 1, to minimize DE , we need to minimize $P'P''$. Also, no matter where you put P , Fact 2 implies that $\triangle P'CP''$ is isosceles, so we want to minimize the base of a family of isosceles triangles. Furthermore, by Fact 3, all these isosceles triangles are similar. So, to find the smallest base, we only need to find the shortest leg. And the legs are always the same length as CP , which is smallest when $\overline{CP} \perp \overline{AB}$.

If you can use a little trigonometry, we can cement in a tighter connection with our special case when $\angle C$ is a right angle.



Let $DE = x$ and $CP = y$. Then $P'P'' = 2x$ and $CP' = CP'' = y$. Use the Law of Cosines on $\triangle CP'P''$ to obtain

$$(2x)^2 = y^2 + y^2 - 2 \cdot y \cdot y \cdot \cos \angle P'CP''$$

$$4x^2 = 2y^2 - 2y^2 \cos 2\angle ACB$$

$$2x^2 = y^2(1 - \cos 2\angle ACB).$$

How nice: If $m\angle ACB = 90^\circ$, then $\cos(2\angle ACB) = -1$, so this says that $2x^2 = 2y^2$, and $x = y$. This is the special case of a right angle. But simplify some more to obtain

$$x = y \sqrt{\frac{1 - \cos(2\angle ACB)}{2}}.$$

Using the sine half-angle formula for angles in quadrants I and II,

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1 - \cos \theta}{2}},$$

we have

$$x = y \sin \angle ACB.$$

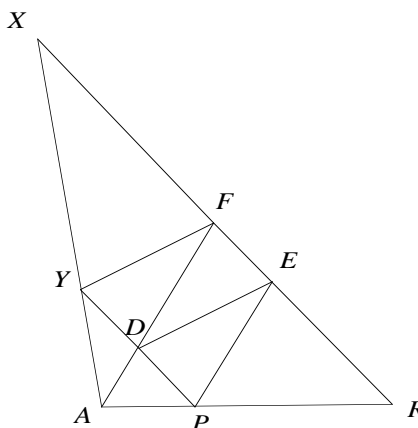
x and y are positive, so taking square roots is allowed.

So, since $\sin \angle ACB$ is a positive constant (it's 1 when $\angle ACB$ is a right angle!), x is a minimum when y is a minimum.

For fun, try graphing x against y on a Cartesian graph.

Problem 14 (Student page 52) This problem is very tricky, and the solution is not at all obvious. In fact, the best location for P might sometimes lie on \overleftrightarrow{AR} but outside of the triangle!

Here's a method for constructing the best location for P : The first step is to double \overline{FR} , producing \overline{FX} . Construct segment \overline{AX} , then extend \overline{PD} , letting Y be the point where it intersects \overline{AX} . Now form \overline{YF} .



Our first goal is to show that $YF = DE$. Then, instead of finding the location for P which minimizes DE , we will find the location which minimizes YF , which will be equivalent.

The problem states that $\overline{PD} \parallel \overline{RF}$ and $\overline{PE} \parallel \overline{AF}$. Since E is on \overline{RF} and D is on \overline{AF} , this tells us that $\overline{PD} \parallel \overline{EF}$ and $\overline{PE} \parallel \overline{DF}$. Therefore, $DPEF$ is a parallelogram, so $\overline{DP} \cong \overline{FE}$, since opposite sides of a parallelogram are congruent. Also,

$$(1) \quad \angle EPD \cong \angle EFD$$

since opposite angles of a parallelogram are congruent.

Since $\overleftrightarrow{DF} \parallel \overleftrightarrow{PE}$, we know that

$$(2) \quad \angle FDY \cong \angle EPD$$

because these are corresponding angles formed by the transversal \overleftrightarrow{YP} and the parallel lines.

From equations (1) and (2) above, it follows that $\angle FDY \cong \angle EFD$. It also follows that $\overline{YF} \parallel \overline{DE}$ because if two lines are cut by a transversal and alternate interior angles are congruent, the lines are parallel. We knew from the given information and the construction that $\overline{YD} \parallel \overline{FE}$. Therefore $YDEF$ is a parallelogram, so

$$YF = DE.$$

Thus, to minimize DE , it suffices to minimize YF . As point P moves, so does Y , but \overline{YF} will always be a path from F to \overline{AX} . The shortest path from a point to a line is always the perpendicular one, so YF will always be smallest when $\overline{YF} \perp \overline{AX}$. Therefore, the best location for P is the spot for which the beginning construction causes \overline{YF} and \overline{AX} to be perpendicular.

Some interesting questions remain: When does P fall outside of \overline{AR} ? What if the original triangle is equilateral? What if it's isosceles? There's a lot of food for thought here.

Problem 15 (Student page 52) At first thought, the answer seems simple: the bigger the diameter of each log in the pile, the greater the total amount of wood in the pile. It seems as if buying larger logs will guarantee that you are buying more wood and less air. Play around with some examples, however. It turns out that, if each log in the pile has the same diameter (and has a circular cross section as in the picture in the *Student Module*), then when you look at the cross section of the 4 ft \times 4 ft \times 8 ft pile you will be looking at a total area of 4π square feet of wood in the pile. This will be true no matter what the diameter of the logs; even if you buy just one large log.

Problem 16 (Student page 53) This problem will be studied extensively in the next section of this module, but experiment with it now if you want to generate a few conjectures.

INTRODUCTION TO THE MAGICAL MIX

This reading investigation provides an introduction to the second section of the module. It contains no problems, so there are no solutions.

A STUDENT OUTSMARTS THE TEST

Problem 2 (*Student page 56*) Rich realized that the desired sum of the distances must not depend on point D 's position inside the triangle. A position was never given, so he decided that he was free to visualize D traveling throughout the interior of the triangle and that, even though D 's position would change, the sum of the distances to the sides of the triangle would stay the same. Rich saw that as he chose a location for D very, very close to a vertex but not quite there, one of the three distances was *almost* the height of the triangle while the other two were extremely small. If the value for the sum of the distances to the sides were the same for *any* point inside the triangle, it must be the same for the points on the triangle itself. So he moved D to a location where the sum would be the easiest to compute—a vertex. It was his reasoning by continuity which convinced him that this would work.

What is reasoning by continuity? Suppose you have a function which acts on the plane—in other words, the function assigns a value to each point in the plane. Reasoning by continuity is the kind of thinking you are doing when you identify the particular function under consideration as one that will assign continuous values (without any gaps or jumps between values), and then use that understanding to interpret the behavior of the values for points that are fairly close together. It is reasoning by continuity that allows us to assume that, given two points extremely close together and a third point between those two, that the value assigned to that third point will also be between the values for the first two points.

Problem 3 (*Student page 56*) The answer is B. Suppose that point D is at vertex C of the triangle, and \overline{CR} is an altitude of $\triangle ABC$. You then have two smaller right triangles, $\triangle CAR$ and $\triangle CBR$. Recall that the altitude of an equilateral triangle bisects the base; therefore, the length of \overline{RB} is 5. Then apply the Pythagorean Theorem to $\triangle CBR$ to find that the answer to the multiple-choice question is B:

$$\begin{aligned}(RB)^2 + (CR)^2 &= (CB)^2 \\ (5)^2 + (CR)^2 &= (10)^2 \\ (CR)^2 &= 75 \\ CR &= \sqrt{75} \\ CR &= 5\sqrt{3}.\end{aligned}$$

Problem 4 (*Student page 57*) The sum of the lengths x , y , and z remains constant, so any individual length will grow or shrink to accommodate the total increase or decrease in the length of the other two.

Problem 5 (*Student page 57*) There are many ways to build a mechanical model of this situation. Here is one fairly simple design:

Use something like coat hanger wire to form an equilateral triangle. You might use three nails as vertices to keep the wire firmly in the correct shape. Tie a piece of string around a large washer and hold the washer inside the triangle. Loop the string around one side of the triangle, and then back through the washer.

Loop the string around a second side, through the washer again, and then around the third side. Pull the string taut through the washer so that the string is perpendicular to the sides. The total length of the string will be *twice* the sum of the distances from the washer to the sides. This length should remain approximately constant regardless of the position of the washer inside the triangle (give or take a little to account for differences in the tautness of the string).

Jiggle the washer and string in moving it around to various positions inside the equilateral triangle so the three loops of string remain perpendicular to the sides. Another version of this model uses a weight attached to the end of the string and a sturdier triangular frame. The weight hangs below the washer and maintains a constant pressure to keep the string taut, but there's still room for some error because you have to hold the washer up, in the same plane as the triangle.

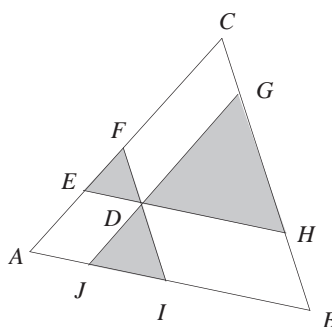
Problem 6 (*Student page 58*) Calculating something in two different ways (in this case the area of the triangle) gives us twice as much information to work with, and in order to create a proof, we'd like to have as much information as possible. Because the area of the triangle is the same no matter how you calculate it, the different expressions for the area may be set equal to each other and reworked according to the rules of algebra. This technique often yields sought-after results.

Another thing to notice about the proof given in the Student Module is that a specific location for D is never used. This is why the proof holds for *any* point inside the triangle—no matter which interior point you use for D , you'll be able to construct three smaller triangles of heights x , y , and z inside the original equilateral triangle.

A proof without words is a real proof, but the reader has to provide the details: the reader has to mathematically justify each step in order to understand the proof.

Problem 7 (Student page 60) Here are the justifications for advancing to Steps 2 through 5:

Step 2: Construct three line segments passing through D , each of which is parallel to one of the three sides of $\triangle ABC$. Notice that this forms three smaller triangles inside $\triangle ABC$ (see the shaded areas in the picture below). Label the points where these new line segments intersect the triangle as E, F, G, H, I, J as in the picture.



The most important fact about this picture, which will be used later, is that the three small triangles, $\triangle DEF$, $\triangle DGH$, and $\triangle DIJ$, are all similar to the large triangle, and hence are all equilateral. Here we show why $\triangle DIJ$ is equilateral (the work for the other two triangles is similar):

Because \overline{AC} is parallel to \overline{JI} , $m\angle DJI = m\angle A = 60^\circ$. ($\angle A$ and $\angle DJI$ are corresponding angles formed when parallel lines are cut by a transversal.) Because \overline{FI} is parallel to \overline{CB} , $m\angle DIJ = m\angle B = 60^\circ$. Now it follows that $m\angle JDI = 60^\circ$, so $\triangle DIJ$ is equilateral.

Step 3: $\triangle DEF$ has been translated upwards to the top of $\triangle ABC$. It fits perfectly since side \overline{FD} was constructed to be parallel to \overline{CG} . $\triangle JDI$ has been translated to the right to fit into the lower right corner of $\triangle ABC$. It also fits perfectly since side \overline{DI} is parallel to \overline{HB} .

Step 4: Here is where we use the fact that the three small triangles are equilateral. In this step, the top two triangles are rotated; the top one is rotated 60° to the left, and the middle one is rotated 60° to the right. Because the triangles are equilateral, it is possible to perform these rotations without changing the positions of the triangles. As a result of these rotations, the three lengths we are interested in are now the vertical altitudes of the three small triangles. An equilateral triangle remains *invariant* upon rotation by 60° .

Step 5: This final step requires noticing that the sum of the lengths of these three altitudes is equal to the length of the altitude of $\triangle ABC$. This means precisely that the sum of the lengths of the perpendiculars from D equals the height of the triangle, which is what we needed to show.

Problem 8 (Student page 61) Draw perpendiculars \overline{WH} , \overline{WI} , and \overline{WJ} from W to the sides of the triangle, where H , I , and J lie on \overline{EF} , \overline{FG} , and \overline{EG} , respectively. Theorem 6.3 tells us that

$$DA + DB + DC = WH + WI + WJ,$$

since each of these sum is equal to the height of $\triangle EFG$.

Next remember that the shortest distance from a point to a line is along a perpendicular path. Therefore $WA > WH$, $WB > WI$, and $WC > WJ$, implying that

$$WA + WB + WC > WH + WI + WJ.$$

Substitute from the equation above to obtain

$$WA + WB + WC > DA + DB + DC.$$

Problem 9 (Student page 61) This question asks you to apply the theorem to a specific example. The answer is $4\sqrt{3}$.

Problem 11 (Student page 62) Triangle ABC is a right triangle with base of length 12 and height 5. Therefore, its area is 30. By applying the Pythagorean Theorem to $\triangle ABC$, we find that $AC = 13$. Now take the base of the triangle to be \overline{AC} . Since $AC = 13$, we know that $30 = \frac{1}{2}(13)(BN)$, so $BN = \frac{60}{13}$. Notice that we have used the technique of calculating area in two different ways again.

Problem 12 (Student page 62) Triangle ABC is a right triangle with area 30. Triangle AST has area

$$\left(\frac{1}{2}\right)(ST)(SA) = \left(\frac{1}{2}\right)(ST)$$

since $SA = 1$. Trapezoid $BSTC$ has area

$$\left(\frac{1}{2}\right)(SB)(ST + BC) = \left(\frac{1}{2}\right)(4)(ST + 12) = 2(ST + 12).$$

Decompose the area of $\triangle ABC$ into two pieces.

$$\text{Area}(\triangle ABC) = \text{Area}(\triangle AST) + \text{Area}(BSTC)$$

$$30 = \left(\frac{1}{2}\right)(ST) + (2)(ST + 12)$$

$$30 = \left(\frac{5}{2}\right)(ST) + 24$$

Thus, $ST = \frac{12}{5}$.

We can also find ST by using similar triangles. (See the *Connected Geometry* module *A Matter of Scale* for a complete discussion of similarity.) Look at $\triangle AST$ and $\triangle ABC$. These two triangles share a common angle ($\angle A$) and each has a right angle, so $\triangle AST \sim \triangle ABC$ by the AA Similarity Theorem. Corresponding sides of similar triangles are proportional, so

$$\frac{AS}{AB} = \frac{ST}{BC}.$$

We are given $AB = 5$ and $SB = 4$, so $AS = 1$. Substitute $AS = 1$, $AB = 5$, and $BC = 12$ in the proportion above:

$$\frac{1}{5} = \frac{ST}{12}.$$

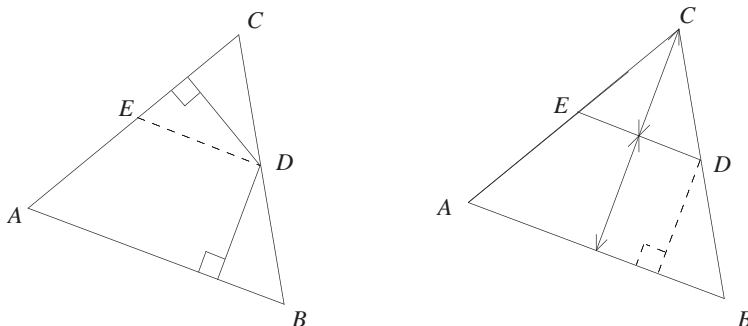
Use cross multiplication to solve for ST :

$$5ST = 12$$

$$ST = \frac{12}{5}.$$

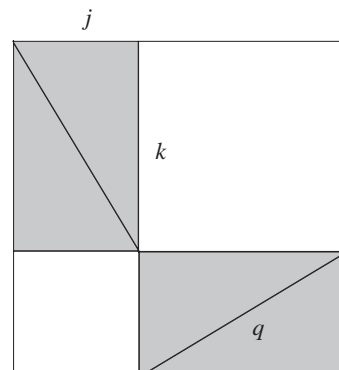
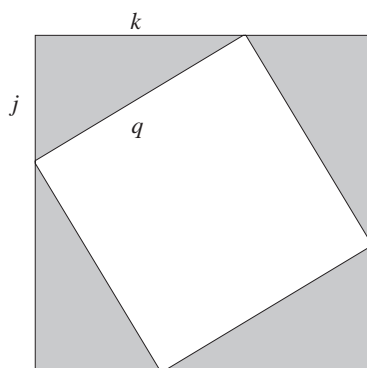
Problem 13 (Student page 62) The proof is similar to the picture proof given earlier in the Student Module in that the key step involves rotating an equilateral triangle. This proof is simpler, though; since D is on the triangle, there are only two perpendiculars instead of three.

Here is the proof:



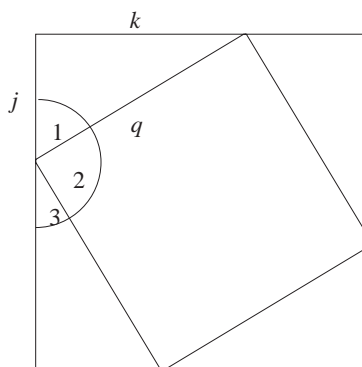
Assume that D is on side \overline{BC} of the triangle, and construct perpendiculars to sides \overline{AC} and \overline{AB} . Draw \overline{DE} parallel to \overline{AB} . Then $m\angle CED = m\angle CAB = 60^\circ$, since the original triangle $\triangle ABC$ is equilateral. Since $m\angle ECD$ also equals 60° , we know that $\triangle ECD$ is equilateral. We then rotate $\triangle ECD$ so that the perpendicular becomes a vertical altitude. It is then clear that the sum of the lengths of the two perpendiculars equals the length of the altitude of $\triangle ABC$.

Problem 14 (Student page 63) Both frames below are squares of sidelength $j + k$. In the original frame, the large square is made up of four right triangles each of area $\frac{1}{2}jk$, plus the interior square of area q^2 . In the second frame, two of the triangles have been translated and rotated in order to line up with the other two triangles. We still have four right triangles of area $\frac{1}{2}jk$, but now are left with two smaller interior squares of areas j^2 and k^2 .



Thus $q^2 + 2jk = j^2 + k^2 + 2jk$, so $q^2 = j^2 + k^2$. Thus, “the sum of the squares of the legs of a right triangle equals the square of the hypotenuse.”

There’s one other interesting problem here: how do we know that the interior quadrilateral really is a square? It looks like it from the picture, but we have to be sure. We know that each side has length q , but we need to verify that each angle is 90° . To check this, label the angles in the drawing as shown below.



Notice that

$$m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ.$$

But angles $\angle 1$ and $\angle 3$ are the two nonright angles of the right triangle, so

$$m\angle 1 + m\angle 3 = 90^\circ.$$

Thus, $\angle 2$ is a right angle. This same argument can be repeated to show that we do indeed have a square.

VARIATIONS ON A
PROBLEM

Problem 1 (Student page 64) Here are some suggestions for alternatives to each of the five assumptions:

- Instead of *the sum of distances*, try: the product of the distances, the maximum distance from a point to a side, or the minimum distance from a point to a side.
- Instead of *from a point*, try: from a side, or from a vertex.
- Instead of *inside*, try: outside, or on the triangle.
- Instead of *an equilateral triangle*, try: an isosceles triangle, a scalene triangle, or a square.
- Instead of *to the sides of the triangle*, try: the vertices of the triangle.

In this investigation, we will look at what happens if point D moves outside of the triangle or if the triangle is not equilateral. In the third section of this module, “The Airport Problem,” we will look at what happens with the sum of the distances to the vertices of any triangle.

Problems 3–4 (Student pages 65 and 68) As point D strays outside the triangle, the value for the function no longer remains constant; as D moves away from the triangle, the values increase.

If you’re using software, you might experience problems with the construction of the perpendiculars. For some locations of D , it is still possible to construct three segments perpendicular to the sides of the triangle. But for other locations of D , the perpendiculars disappear. To solve this problem, connect D perpendicularly to the *line* through that side of the triangle. Then you’ll be able to construct a perpendicular segment, although it will lie outside of the triangle. This changes the problem in some sense, but the flavor of the original problem hasn’t changed.

Problem 5 (Student page 68)

- a. The function is not constant on the exterior of the triangle. The farther away you get from the sides of the triangle, the larger the values of the function become. This makes sense. What is surprising, however, is that there are regions in which the function does remain constant.
- b. There aren’t any places outside the triangle that produce a smaller value than the value on the interior. The minimum value for the function is the constant value it obtains as D moves throughout the interior.
- c. Since the sum of the distances continues to grow as D moves farther from the triangle, there is no maximum value for the function.

We know this through reasoning by continuity, not because we can check the value for every point outside the triangle.

Software may give you a good idea of where the values increase for D , but you may find it easier to sketch the contour lines on paper by approximating points of constant value with a ruler.

Can you give a more formal description of a contour plot?

- d. The constant regions, or contour lines, are hexagons surrounding the triangle, three sides of which are parallel to the sides of the triangle. For a detailed analysis of this see the solution for Problem 10.

Problem 7 (Student page 69) Recall that a contour line is a set of points whose assigned values from a function are all the same. A contour plot is a collection of contour lines, all drawn in the same picture, with information given to let you estimate the different values of the function. A surface plot is a three-dimensional drawing where each contour line is lifted up to its appropriate height.

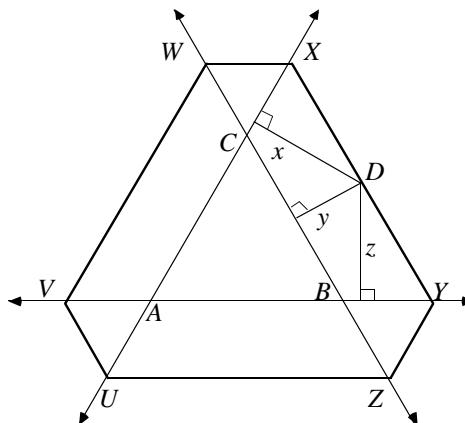
The difference between contour plots and surface plots is the difference between topographical maps and relief maps. If you have a surface plot and would like to picture the corresponding contour plot, one way is to imagine pushing the surface plot down and flattening it out. The resulting lines would form a contour plot. Similarly, if you start with a contour plot and imagine lifting the lines up to their assigned heights, you will be visualizing the surface plot (or the outline of it).

Suppose again that you have a surface plot. If you take the surface and slice it with a plane parallel to the floor, the outline of the cross section will be a contour line. If you slice the surface at a variety of different heights, you will have a contour plot. Or, if your surface plot happens to be concave upward, you can imagine it as a tub filled with water. If you fill the tub up to different heights, when projected to the floor the resulting water marks will give a contour plot.

Problem 8 (Student page 69) The surface plot corresponding to the dartboard would look like an extended telescope or a stack of cylinders of ever-decreasing radii. Whether the telescope is standing on its smallest or largest cylinder would depend on whether you imagined the contour lines of the dartboard being lifted up or pushed down.

Problem 9 (Student page 69) The important idea here is to convey the concept of a series of different hexagons surrounding the triangle at different distances away from it. Opposite sides of the hexagons are parallel, and they are parallel to a side of the triangle. One contour line is pictured in the solution for Problem 10 on the following page.

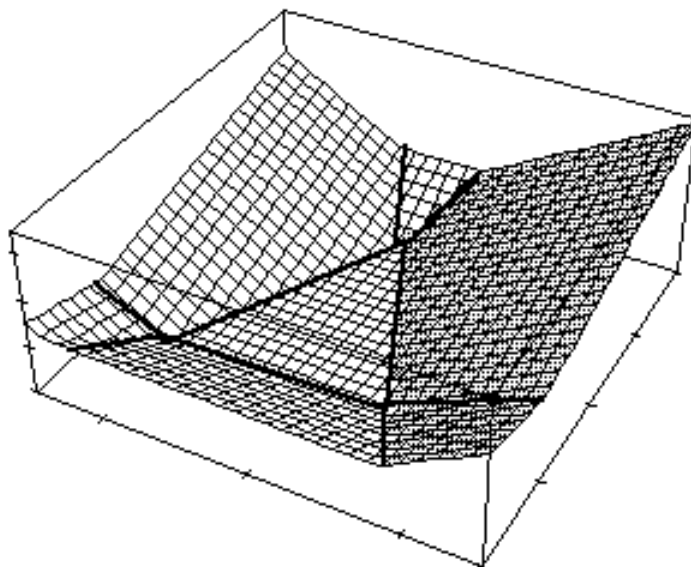
Problem 10 (Student page 69) One way to do this problem makes nice use of geometry and reasoning by continuity. You want to show that the sum of the distances is constant on a hexagon like this:



First show that the sum is constant along \overline{XY} . To do this, look at $\triangle XAY$. Show that it's isosceles (indeed, it's equilateral), and then use an area argument to show that $x + z$ is constant as D moves back and forth on \overline{XY} . (Break the area of $\triangle XAY$ into the sum of the areas of $\triangle XAD$ and $\triangle YAD$. The first of these two triangles has base \overline{XA} and height x . The second of these two has base \overline{YA} and height z . But $XA = YA$. When you write it all out, you find that $x + z$ is always equal to the height from Y to \overline{XA} .) We also know that y is constant as D moves along \overline{XY} because \overline{BC} is parallel to \overline{XY} , so we have shown that the sum $x + y + z$ is constant.

Similarly, show that $x + y + z$ is constant on the other sides of the hexagon (the argument is exactly the same on the long sides and almost the same on the short sides). Now all you have to do is to show that these constants all agree at the corners of the hexagon. That's where the continuous nature of this function comes in. If you approach X along \overline{WX} , the value of Rich's function should get closer to the value of the function as you approach X along \overline{XY} . So the value can't be 10 along \overline{WX} and then suddenly 12 along \overline{XY} .

Problem 11 (Student page 69) The surface plot for Rich’s function looks like this:



It was generated in Mathematica®. There are seven planar regions. The middle one is directly over the equilateral triangle (it is parallel to the triangle and is h units above it, where h is the height of the triangle). Imagine that this is a tub filled with water. What would the water lines look like? If the water lines are projected onto the floor, you get the contour lines. (The bottom of the tub would be the equilateral triangle; above that, the water lines at any level will be hexagons.)

Problem 12 (Student page 69) It doesn’t make sense to say “the sum of the distances to the sides of a nonequilateral triangle will always be equal to the height of the triangle,” since a nonequilateral triangle has no one distinct height! It depends from which vertex you measure the height.

A student might hear this and ask, “How can you talk about the area of the triangle, then, since area is $\frac{1}{2}$ base times height?” This is why the area formula is so nice; it shows that, although we may get different heights from each vertex, these heights are related to the lengths of the corresponding bases, and this relation involves the area of the triangle.

Problem 13 (Student page 70) The reasoning in the solution for Problem 12 does not indicate that the function that calculates the sum of the distances to the sides can't be constant on the interior of a nonequilateral triangle. But it does point out that the function can't be constant and equal to the height of the triangle, since that height is not well-defined. The function could still be constant but equal to some other value.

Problem 14 (Student page 70) Make sure you consider “special” triangles, such as isosceles or right triangles, as well as acute and obtuse scalene triangles. It's important to see that this is not a completely new problem, but rather a generalization of Rich's problem.

The behavior of this function is not constant on the interior of the nonequilateral triangle. You may have noticed that often just one of the vertices seems to be the spot where the function has a minimum value but that this is not *always* true. (What do the heights of the various triangles have to do with this?)

Problem 15 (Student page 71) To see what goes wrong if we try to adapt the two proofs we used for the equilateral triangle, start by mimicking each proof. This new triangle can still be divided into three smaller triangles: $\triangle ACD$, $\triangle CDB$, and $\triangle ADB$. Their bases are \overline{AC} , \overline{CB} , and \overline{AB} , respectively. The problem is that these small triangles no longer have bases of the same length because $\triangle ABC$ isn't equilateral.

The algebraic proof we used in Problem 6 of Investigation 6.7, however, relies on these three triangles all having congruent bases with lengths called b . The proof fails at this point.

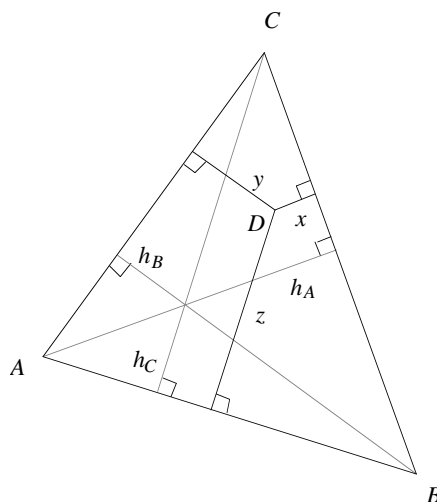
Similarly, when $\triangle ABC$ is not equilateral, the picture proof from Problem 7 of Investigation 6.7 also fails. The three smaller triangles formed will each fail to be equilateral, and the rotation step of the proof will not be possible.

Problem 16 (Student page 72) The function takes on the smallest value when D is at the vertex opposite the longest side (the vertex of the largest angle). It takes on the largest value when D is at the vertex opposite the shortest side (the vertex of the smallest angle).

Problem 17 (Student page 72) The minimum and maximum values for the function are equal to the lengths of the shortest and longest altitudes of the triangle. Actually, the function can never take on those values, since points *on* the triangle are

not *in* the triangle. But you can never get a value less than this “minimum” or more than this “maximum.” How close can you get to these values?

Suppose we let h_A , h_B , and h_C be the lengths of the altitudes of the triangle from vertices A , B , and C , respectively. (See the picture below.) Assume that $h_A \leq h_B \leq h_C$. This means that $m\angle A \geq m\angle B \geq m\angle C$, and also that \overline{AB} is the smallest side of $\triangle ABC$, while \overline{BC} is the largest side. Also, let x , y , and z be the distances from D to the sides of the triangle.



We can try to prove the above statements by trying to mimic the earlier proof as much as possible. Let's once again calculate the area of $\triangle ABC$ two different ways:

$$\begin{aligned}\text{Area}(\triangle ABC) &= \text{Area}(\triangle ADB) + \text{Area}(\triangle CDB) + \text{Area}(\triangle ADC) \\ \frac{1}{2}(AB)h_C &= \frac{1}{2}(AB)z + \frac{1}{2}(CB)x + \frac{1}{2}(AC)y \\ &\geq \frac{1}{2}(AB)x + \frac{1}{2}(AB)y + \frac{1}{2}(AB)z \\ &= \frac{1}{2}(AB)(x + y + z)\end{aligned}$$

This means that $h_C \geq x + y + z$.

Notice what we did above: We started out the same way as in the earlier proof, but when we ran into difficulty (all sides not being the same length), we substituted an inequality for the equality and continued along. Similarly, we can use the fact that \overline{BC} is the largest side to show that $h_C \leq x + y + z$. This proves that the minimum and maximum values of the function are the lengths of the shortest and longest altitudes of the triangle.

Are these the only places where $x + y + z = h_C$ or $x + y + z = h_A$?

This is different from the case of an isosceles triangle, where you may have two vertices sharing the minimum value or two sharing the maximum.

The function achieves its maximum value when $x + y + z = h_C$. If D is at the vertex C , then $y = x = 0$ and $z = h_C$, implying that $x + y + z = h_C$. Also, if D is at the vertex A then $z = y = 0$ and $x = h_A$, implying that $x + y + z = h_A$ and the function is at its minimum value.

Problem 18 (Student page 72) When the minimum and maximum values of the function on the interior of a triangle are equal, all altitudes of the triangle will have the same length, and hence the triangle is equilateral. Further, if the maximum and minimum values of the function are equal, then the function is constant. This is precisely what we knew already, that the function is constant on an equilateral triangle.

Problem 19 (Student page 72) We can model this on the situation in the last problem, but assume that $z = 0$, for this is exactly what it means for D to lie on \overline{AB} .

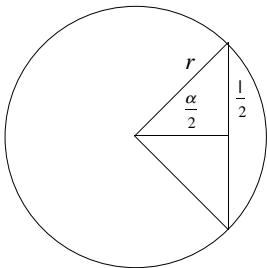
Therefore, the results are that the smallest value of the function is the minimum of h_A and h_B , while the largest value is the maximum of h_A and h_B . The function achieves its maximum value when D is at the vertex of the the smaller of A and B , because the larger altitude is from the smaller angle. Furthermore, the function achieves its minimum value when D is at the vertex of the larger of $\angle A$ and $\angle B$, because the smaller altitude is from the larger angle.

One point about this problem is a bit subtle. The function is still constant when the triangle is equilateral, but now it is also constant when the triangle is isosceles and segments \overline{AC} and \overline{BC} are congruent.

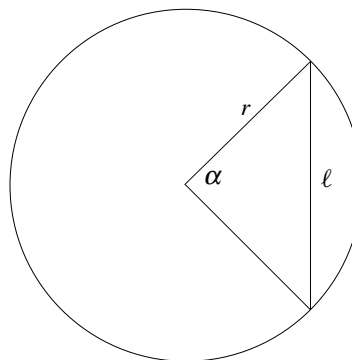
Problem 20 (Student page 73)

- a. This is the same as Problem 13 of Investigation 6.5. The position for D that minimizes the value of the function is at the foot of the altitude from C . When we presented a solution there, we promised you another one. Although this one uses a bit of trigonometry, we thought you might enjoy it.

Hint:



First, convince yourself that, if a circle has radius r and diameter d , and if its chord has length ℓ and central angle α , then $\ell = 2r \sin \frac{\alpha}{2} = d \sin \frac{\alpha}{2}$.

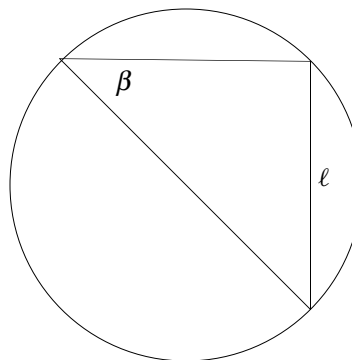


$$\ell = d \sin \frac{\alpha}{2}$$

Since the measure of an inscribed angle is half the measure of the corresponding central angle, we see that if an inscribed angle β intercepts a chord of length ℓ , then

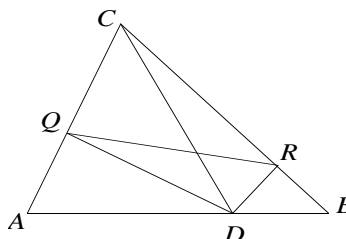
$$\ell = d \sin \beta,$$

where d is the diameter of the circle.



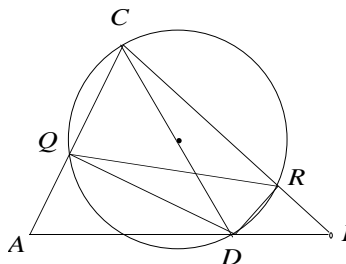
$$\ell = d \sin \beta$$

Now, back to the situation at hand.



Why is this?

Since triangles $\triangle DQC$ and $\triangle DRC$ are right triangles, there is a circle with diameter \overline{CD} that passes through Q , D , R , and C :



Applying the equation $\ell = d \sin \beta$ to chord \overline{QR} , diameter \overline{CD} , and inscribed angle $\angle ACD$, we obtain

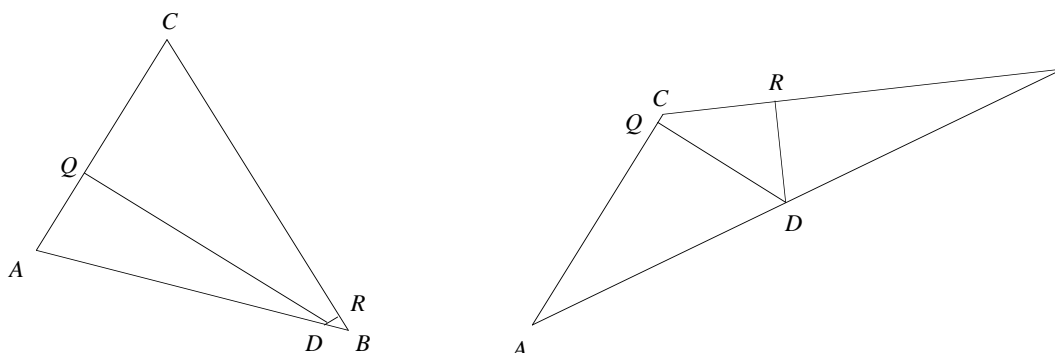
$$QR = CD \sin \angle ACB.$$

But $\sin \angle ACB$ is *constant* (because $\angle ACB$ is), and QR is some (positive) constant multiple of CD . Therefore, QR is smallest when CD is. But CD is smallest when $\overline{CD} \perp \overline{AB}$. That is, when D is at the foot of the altitude from C to \overline{AB} .

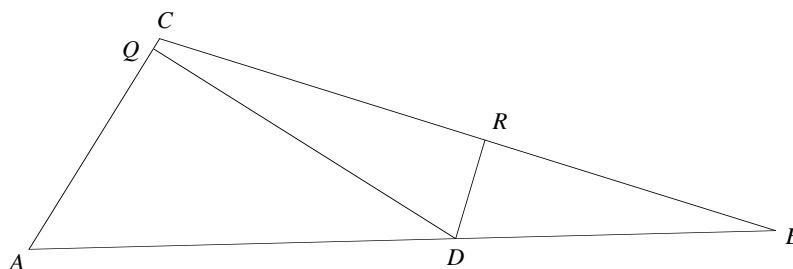
Notice that this also gives the minimum *value* for QR , $h \sin \angle ACB$, where h is the height from C to \overline{AB} .

- b.** The position for D that yields a maximum value for QR depends upon the size of the angle at C . If $m\angle ACB = 90^\circ$, the maximum value is the length of the longer of the two sides (either \overline{AC} or \overline{CB}). If $\triangle ABC$ is a nonright, nonisosceles triangle, the position for D that will produce the maximum

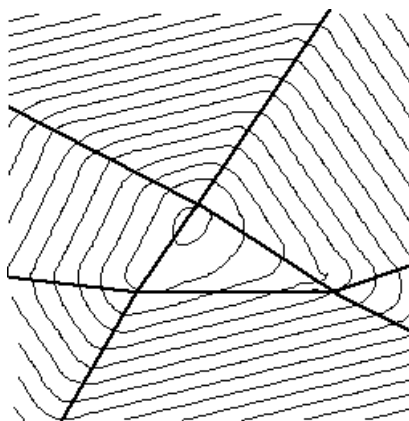
value for QR is the spot that's as close as it can get, while maintaining the perpendiculars to the sides, to the vertex of the longer of the two sides. (An isosceles triangle will have two such points.)



When the angle at C is obtuse or acute, the maximum length of \overline{QR} can't come close to the length of the longer side because either a perpendicular threatens to extend beyond the boundary of the triangle or D reaches the vertex of the longer side.



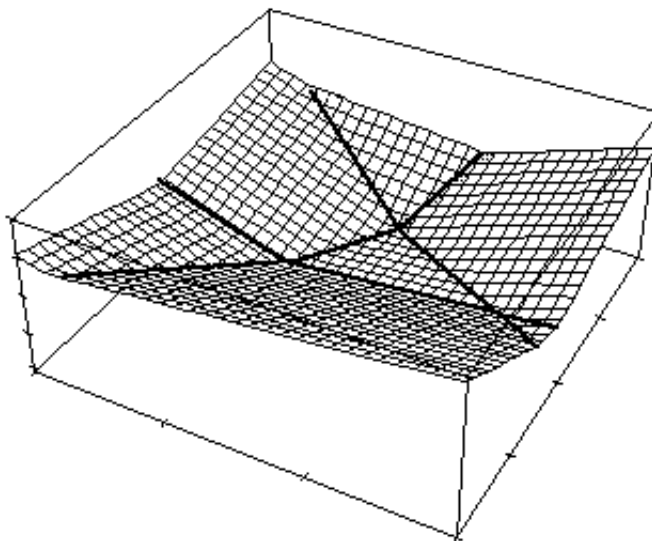
Problem 21 (Student page 74) One way to do this is by using the slider point method, described in detail in the *Teaching Notes* for Investigation 6.4. Another way is to use the built-in **Contour-Plot** primitive in Mathematica. That produces a picture like this:



The curvy corners in the contour lines are due to the inaccuracies of the software. One can prove that they are, in fact, straight lines (see the *Teaching Notes* for Investigation 6.4).

Problem 22 (*Student page 74*) Try to visualize (and then sketch) the surface plot for the function over a nonequilateral triangle by looking at the contour lines drawn for the solution to Problem 21 and by referring back to the solution to Problem 20. Three points that might be easiest to start with are the vertices of the triangle. In the surface plot, each of these vertices will be located h units directly above the corresponding vertices of the original triangle. At each of these points, the value for h will be equal to the length (in the original triangle) of the altitude from the corresponding vertex to the opposite side. So the planar region that contains these three points and sits above the original triangle will not be parallel to the original triangle—it will slant upward.

A surface plot generated in Mathematica shows all seven planar regions:



It's not easy to see that the regions in the surface plot will be planar. A proof that they are indeed planar may be obtained by writing to the authors.

Start at the lowest vertex of the nonequilateral base of this very funny tub, and imagine tracing a few contour lines around it. You get quadrilaterals. What else do you get as you trace out contour lines of greater values?

If you were asked to imagine that this is a tub filled with water, could you describe what the water lines would look like? When the water lines are projected onto the floor, you get the contour lines. What kinds of polygons can show up as contour lines?

MAKING CONNECTIONS IN MATHEMATICS

Problem 1 (*Student page 78*) This is a fictional version of something that really happened with a telephone company. Rather than hook up each office to the other two so that the network would form a triangle, the company connected two of the offices through the third, choosing the shortest two sides of the triangle. All communication routed through that “middle” office, until one day, when a client realized that having four offices could incur less in telecommunication costs than the company’s setup did for three.

Problem 2 (*Student page 78*) There are many spots inside the triangle formed by New York City, Miami, and Kansas City for which the sum of the distances to the three cities is less than the smallest sum of the lengths of any two sides of the triangle.

Problem 3 (*Student page 78*) There is one spot inside the triangle which minimizes the sum of the distances—here you might want to focus on the “why” rather than the “where,” which is the focus of the third section of this module, “The Airport Problem.” Reasoning by continuity tells us that the contour lines for this function will shrink down to a line segment or a point.

USING THE MAGICAL MIX

Problem 1 (Student page 79)

- a. $GJ = FJ$ because \overline{HJ} is a median.
- b. GJ ng GH
- c. $FK = FJ$. Since \overline{GK} and \overline{HJ} are medians, $FK = \frac{1}{2}FH$, and $FJ = \frac{1}{2}FG$. But $FH = FG$, so $FK = FJ$.
- d. $FJ + JH > FK + KH$. Both $FJ + JH$ and $FK + KH$ are paths from F to H , but $FK + KH$ is shorter, as it is a straight line segment.
- e. $\text{Area}(\triangle GJH) = \text{Area}(\triangle HKG)$. The two triangles are congruent, so their areas are equal.
- f. $m\angle KHG > m\angle JGK$. We know $m\angle KHG = m\angle JGH$ because $\triangle FGH$ is isosceles. It is clear that $m\angle JGK < m\angle JGH$, so we know that $m\angle JGK < m\angle KHG$.
- g. $\text{Area}(\triangle FKG) = \text{Area}(\triangle HKG)$. Draw the altitude from G to \overline{FH} . Let P be the foot of this perpendicular. Observe that \overline{GP} is an altitude of both triangles. (For obtuse triangle FKG , this is the altitude that falls outside the triangle.) Since $FK = HK$ (because \overline{GK} is a median), $\triangle FKG$ and $\triangle HKG$ have equal-length bases and the same altitude, so they must have the same area.
- h. The distance from G to K is greater than (or perhaps equal to) the distance from G to \overline{FH} . The shortest distance from a point to a line is along a perpendicular path; that is what is meant by the distance from G to \overline{FH} . Point K is on \overline{FH} , yet \overline{GK} does not intersect \overline{FH} perpendicularly. Therefore, GK is greater than the distance from G to \overline{FH} .
- i. $JK < GH$. The base \overline{GH} is the “widest” part of the triangle.

The distances will be equal only if the triangle is equilateral. In an equilateral triangle, every median is also an altitude, so the distance from G to \overline{FH} would be GK .

Problem 2 (Student page 79) One way to approach this problem is to study the reasoning used in the solution for each part of Problem 1 and see where the assumption that the triangle is isosceles was used. If it wasn’t used, then the same reasoning will still hold. This is another chance to emphasize the importance of studying proofs to see where things go wrong if certain assumptions are changed or relaxed.

- a. $GJ = FJ$. This is the same, by the definition of a median.
- b. GJ ng GH . This is the same.
- c. FK ng FJ . This is changed to an ng because now $FG \neq FH$, but we don’t know which is larger.

- d. $FJ + JH > FK + KH$. This is the same; in Problem 1d, the fact that the triangle was isosceles was never used.
- e. $\text{Area}(\triangle GJH) = \text{Area}(\triangle GHK)$. These two triangles are no longer congruent, but they still have equal areas: First, the triangles share a common base, \overline{GH} . Since J and K are still midpoints, the Midpoint Theorem tells us that $\overline{JK} \parallel \overline{GH}$. Therefore, the perpendicular segments from J to \overline{GH} and from K to \overline{GH} will have the same length. This means that the triangles have the same base and height, so their areas must be equal.
- f. $m\angle KHG$ and $m\angle JGK$. Again it's changed to an ng because the solution for Problem 1f required that the two sides of the triangle were congruent.
- g. $\text{Area}(\triangle FKG) = \text{Area}(\triangle HKG)$. This is the same as in Problem 1g since \overline{GK} is still a median.
- h. The distance from G to K is greater than the distance from G to \overline{FH} . This is the same as in Problem 1h except that here the distances cannot possibly be equal.
- i. $JK < GH$. This is the same as in Problem 1i.

Problem 4 (Student page 80) Equilateral triangles have many interesting properties. You may think of some more that are not listed here.

1. Every equilateral triangle has three congruent sides (by definition). This is not true for any other kind of triangle.
2. Every equilateral triangle has three congruent angles, each measuring 60 degrees. This is not true for any other kind of triangle.
3. All equilateral triangles are similar. This is not true of all isosceles, scalene, right, acute, or obtuse triangles.
4. All medians of an equilateral triangle are congruent. In an isosceles triangle, two of the medians are congruent, while in a scalene triangle none of the medians are congruent.
5. All altitudes of an equilateral triangle are congruent. In an isosceles triangle, two of the altitudes are congruent, while in a scalene triangle none of the altitudes are congruent.
6. All angle bisectors of an equilateral triangle are congruent. In an isosceles triangle, two of the altitudes are congruent, while in a scalene triangle none of the angle bisectors are congruent.

7. In an equilateral triangle, the median, altitude, and angle bisector drawn from any chosen vertex are all the same segment. Such a segment divides the equilateral into two congruent 30–60–90 triangles. This property is only true for equilateral triangles.
8. Each of the segments mentioned in item 7 above forms a line of symmetry for the triangle. This property is only true for equilateral triangles.
9. The area of an equilateral triangle of sidelength s is given by the formula

$$A = \frac{s^2\sqrt{3}}{4}.$$

This is a special formula for equilateral triangles only.

10. If an equilateral triangle is rotated 120 degrees either clockwise or counterclockwise about its center, the new triangle will coincide exactly with the original one. Other kinds of triangles do not have this property.
11. Of all triangles with the same perimeter, the equilateral one has the greatest area.

Problem 5 (Student page 80) A reasonable definition might be, “a function is a rule which assigns a unique value to each element of some set.” A rule which assigns to every number its square roots isn’t a function because this rule assigns more than one value to each number.

Problem 6 (Student page 80)

- i. The measure of $\angle APB$ is constant and always equal to 90° since $\angle APB$ is inscribed in a semicircle.
- ii. The distance from P to O is always 2 since \overline{OP} is always a radius of the circle.
- iii. The perimeter of $\triangle APB$ changes. This one is tricky. We know that AB stays fixed, as it is the diameter of the circle, so we just need to look at what happens to $AP + PB$. Let $x = AP$ and $y = PB$. Then it turns out that the quantity $x + y$ varies between the values 4 and $4\sqrt{2} \approx 5.66$. When P is at point A , $x + y = 4$. The value then increases until P is at the point on top of the circle where \overline{PO} is perpendicular to \overline{AB} ; here the

If you are familiar with the Law of Cosines from trigonometry, you can use it to actually solve for x and y in terms of each other, and use that to see how the quantity $x + y$ varies.

value of $x + y$ is $4\sqrt{2}$. Then the value decreases, reaching 4 again when $P = B$. It then increases until P is at the bottom of the circle, where \overline{PO} is perpendicular to \overline{AB} and then decreases again until $P = A$.

- iv. The area of $\triangle APB$ changes. If P starts at A , where there is no triangle, the area is 0. It increases until P gets to the top of the circle, where the area is 4, and then decreases again as P gets closer and closer to B . It then increases again until P gets to the bottom of the circle and then once again decreases until P is back to A .
- v. The ratio of the circumference of a circle to the diameter of a circle is constant and equal to π .
- vi. The sum of the distances $AP + PB$ changes. This is exactly what we analyzed when discussing the perimeter in part iii.
- vii. The ratio of AP to BA changes. The value BA is constant, as it is the diameter of the circle. As P moves from A to B , this ratio increases from 0 to 1. As P continues, moving from B back to A , the ratio decreases from 1 to 0.
- viii. The ratio of AP to PB changes. As P moves from A to B , AP increases while PB decreases. The ratio thus increases. As P moves from B to A , PB increases and AP decreases, causing the ratio to decrease.
- ix. The distance from M to N , where M is the midpoint of \overline{AP} and N is the midpoint of \overline{PB} , never changes since $MN = \frac{1}{2}AB$ by the Midline Theorem, and AB is constant.

GETTING STARTED

What do you think of this idea? Put the cities on a coordinate plane. Each city is now an ordered pair. Scale each pair by the city's population, add the resulting points, and divide by 3. In what sense is this "fair"?

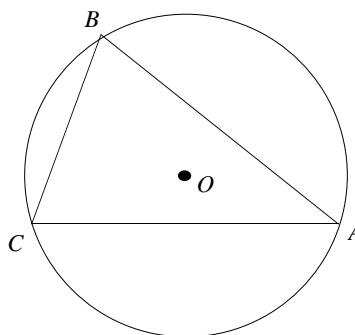
Problem 1 (*Student page 82*) The importance of choosing a location for the airport that is either fair to the people in all three cities, or that is environmentally preferable, is discussed in later questions. Here are some other points to keep in mind. When looking at maps to choose a location, be aware of local geography. What if the spot that looks equidistant from all three cities happens to be in the middle of a lake or at the top of a mountain? The relative size of the cities is important, also. If the majority of the travelers using the airport will come from one of the cities, it probably makes sense to place the airport nearer to that city. Another point of interest might be to look for a place where there is enough undeveloped land available.

WHAT IS MEANT BY BEST?

Problem 1 (Student page 83) There are many mathematical definitions of “best” which could be adopted: finding the point equidistant from all three cities, finding the point which minimizes the sum of the distances to the cities, or perhaps finding the point on some fixed line which minimizes the sum of the distances. In terms of what should be recommended to a city council, keep in mind what might be important to the people living in the cities. In other words, the mathematics is very important, but in real-life situations there are often other considerations; the very best mathematical solution may not be socially ideal.

There’s an important distinction to make here: there’s a solution to the mathematical problem of finding a point subject to some well-defined constraints, and there’s a solution to the much more complex socio-political problem of locating an airport. While the former can inform the latter, these problems are *different*. Extra-mathematical considerations are crucial in political considerations, but they are not at all part of the mathematical investigation.

Problem 2 (Student page 83) The “fairness” solution to the airport problem amounts to finding a spot that is equidistant from all three cities. This means finding the center of a circle that passes through each of the cities:



A circle around $\triangle ABC$

If the airport is at the center O , then the distances from the airport to each of the cities will be equal to the radius of the circle. This circle is the *circumcircle* for the triangle, and its center is the triangle’s *circumcenter*. The easiest way to construct it is to find the intersection of the perpendicular bisectors of any two sides of the triangle. This construction works because of the Perpendicular Bisector Theorem: A point is on the perpendicular bisector of a segment if and only if it is equidistant from the endpoints of the segment. This theorem was discussed in Investigation 6.2.

Problem 3 (Student page 83) Here are some possible ideas of situations where the fairness solution might not be the best place to put the airport:

- If two cities are very close and the third is far away, the fairness solution would be an inconvenience for the two neighboring cities.
- Likewise, if the triangle formed by the three cities has a “very obtuse” angle (an angle much larger than 90°), the fairness solution may unnecessarily force everyone to drive a relatively long distance to the airport.

Problem 4 (Student page 83) The environmental solution minimizes the amount of new roadway which would have to be built. Mathematically, this means minimizing the *sum* of the distances from the airport to each of the three cities. The fairness solution, on the other hand, amounts to finding the center of the circle passing through the three cities. For most triangles, these two solutions are very different. Sometimes, however, they do occur at the same spot in particular triangles. For the moment, it is left to you to conjecture or explain in which triangles that might occur. Later on, you may even prove why they coincide in particular cases.

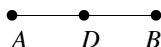
Problem 5 (Student page 85) Think back to the function examined in the second section of this module (Rich’s function): find the point inside a triangle where the sum of the distance from the point to the sides of the triangle is minimized. This is a perfect example of a continuous system. As point D moves inside the triangle, you calculate the sum of the distances at each location and observe how the value changes in relation to the change in the location of D .

Problems 6–8 (Student pages 85–86) These problems are meant as opportunities for you to experiment and to come up with conjectures for the question of where to put the airport. We won’t spoil the fun by providing answers just yet. In fact, the investigation that follows focuses solely on different ways of analyzing this system. We hope the problems there provide enough examples of different ways of thinking about it to get you started and to give you the time and experience needed to develop and support a conjecture about the best spot.

SPECIAL CASES AND MODELS

If D is between A and B ,
 $AD + DB = AB$. If D is *not*
between A and B ,
 $AD + DB > AB$ by the
Triangle Inequality.

When we also want to
consider the fairness
question, then the best
place for D would clearly be
midway between A and B .



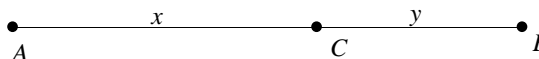
Draw a few pictures of
locations for D or imagine
them in your mind.

This is an example of the
most economical solution
not necessarily being the
fairest — the people living
in City C won't have to
spend any money on roads
to the airport. On the other
hand, an airport in City C
may not make the people
who live in City C very
happy. Is there a way to
keep everyone relatively
happy?

Problem 1 (Student page 87) If given only two cities, say A and B , the problem is reduced to finding a point D so that the sum $AD + DB$ is as small as possible. If D is anywhere on \overline{AB} (between or at the endpoints A and B), it will always be true that $AD + DB = AB$. Any other location for D will produce a larger value: anywhere else on the line containing A and B will be bad; if D is not on that line, then $AD + DB$ will be a path from A to B which is not a straight line segment—the value $AD + DB$ will be greater than AB . Thus, the total minimum distance to the airport is the length of \overline{AB} .

Thus, for the environmental (or economical) solution, there are many spots: all of the points on \overline{AB} .

Problem 2 (Student page 87) Suppose the cities are laid out like this on a line segment.



If D is still the roving airport looking for a home, we'll want to minimize $AD + DC + DB$. We know from the Triangle Inequality that

$$AD + DC \geq AC,$$

with equality occurring only when D is between A and C . Thus

$$AD + DC + DB \geq AC + DB,$$

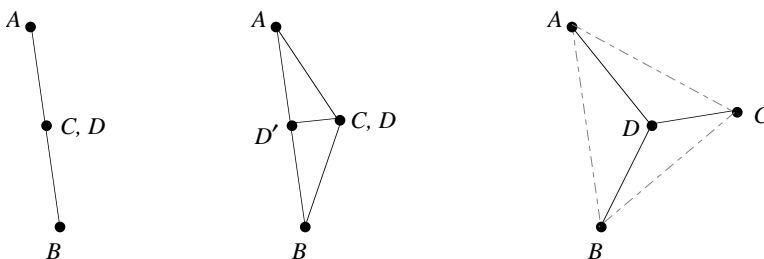
with equality occurring, again, when D is between A and C . So, to minimize the sum, D must be between A and C . But if D is so placed,

$$AD + DC + DB = AC + DB \geq AC + CB,$$

with equality occurring when $D = C$. Hence, the absolute minimum happens when $D = C$, and the minimum sum is $AD + DB = AB$.

This problem is best done experimentally using software that can measure the tiny differences in distance. Theoretically, it's about whether, for a right triangle with another angle between 60° and 90° , the Triangle Inequality can be amended to say that the length of the hypotenuse will be less than the sum of the length of the longest leg and *half* the length of the shortest leg.

Problem 3 (Student page 87) Now the question becomes whether D , in representing the best environmental solution, will remain on \overline{AB} , as close as possible to C , or whether it will move off with C as the location for C moves just off \overline{AB} . As it turns out, D does go with C as long as C is *almost* collinear with A and B . (See the figure below.) We see that $AD + BD$ (where $CD = 0$) $< AD' + CD' + BD'$, where D' is located at the previous spot for C on \overline{AB} . So D sticks with C at first, but clearly separates from C somewhere, as you will see in the case of equilateral triangle ABC in Problem 5, where D is located in some kind of center for the triangle. The question is, when does D separate, and does it still get pulled along by C , only more slowly, or not at all?



At first, the airport (D) is at the same place as C . Then the airport moves with C , while

$$AD < AD' + \frac{1}{2}CD'.$$

Finally, the airport (still shown by D), separates from C , and stays inside $\triangle ABC$.

Problem 4 (Student page 88) The best location to minimize total distance to the airport is quite a bit closer to \overline{AB} than shown in the picture in the Student Module. So when one city is very far away from the other two, the position for D is relatively near the two cities that are close to each other.

Reasoning by continuity, this means that because D at first moves off of \overline{AB} along with C (as C moves constantly farther and farther away from \overline{AB}), that D will stay with the location for C until some point at which it either: 1) stops while C keeps going; or 2) begins to move only part of the distance C moves. As you experiment with this some more, you will find that, no matter how far out City C is placed, at some point the airport location just doesn't move at all. This argues for the first of the two hypotheses we have just made.

Take the example given in the Student Module for this problem: If nothing else changes except that City C is twice as far from \overline{AB} as shown in the figure, the best

position for the airport at D does not change. Thus, C 's distance from \overline{AB} must not be the primary player in this system. If not distance, what other relationship between A , B , and C might be the determining factor for the location of D ?

Problem 5 (Student page 89) In an equilateral triangle, the airport does indeed go in the “center”—that’s the spot that minimizes total distance to the vertices of the triangle. If you start with three collinear cities, where one city lies at the midpoint between the other two, and you gradually move the middle city out along the perpendicular bisector of the segment between the other two cities, then the location for the airport “leaves” the middle city at this particular spot, which is some kind of center of an equilateral triangle. Furthermore, the location for the airport separates from that middle city long *before* the three cities form, and even *after* they no longer form an equilateral triangle.

Problem 6 (Student page 89)

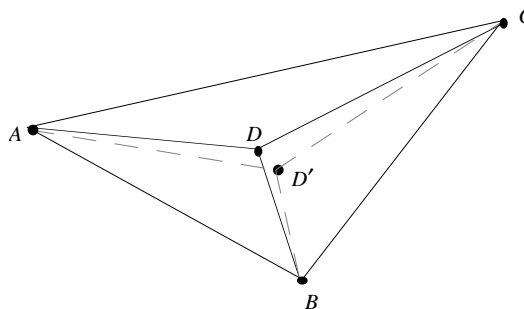
- a. To locate the spot for the airport that is a minimum total distance from the three cities, you find the place which makes angles of 120° with each pair of cities. This coincides with the various centers of an equilateral triangle, and is the “Fermat point” for nonequilateral triangles. You can find this spot by trial and error or by devising some sort of pre-fabricated 120° point on translucent paper or with wire, and placing this on top of the triangle. Some triangles don’t have this 120° point on their interior. Then the spot that minimizes total distance to the three vertices is located at the vertex with the largest angle.
- b. Besides coming up with a proof that this is the spot that minimizes distance, one way to convince people might be to use a reasoning-by-continuity argument. Start with line segments of various lengths, and work through the argument that, while C is collinear or “nearly collinear” with A and B , the spot for the airport remains at C , but as soon as $\angle ACB$ is greater than 120° , the spot for the airport falls inside the triangle.

Problem 7 (Student page 90) This is another example of reasoning by continuity. Here is the idea: Suppose you have a point D , but then you move the point to a new location, which is just a little bit closer to B . The distance to B has decreased a bit, but what has probably happened is that the distances to A and C have increased a little. Thus, the net effect is that the sum of the distances changes very little when D is moved only a small distance.

You can read more about the Fermat point in Investigation 6.15.

This argument does require an awful lot of examples of different triangles. At some point, a proof will be needed to really convince people.

This can be demonstrated using the Triangle Inequality. Suppose the point D is moved slightly to the point D' ; by “slightly” we mean that the distance DD' is very small. We want to look at $AD' + BD' + CD'$ and compare it to $AD + BD + CD$.



The Triangle Inequality says that

$$AD' \leq AD + DD'$$

$$BD' \leq BD + DD'$$

and

$$CD' \leq CD + DD'.$$

Therefore,

$$\begin{aligned} AD' + BD' + CD' &\leq (AD + DD') + (BD + DD') + (CD + DD') \\ &= 3DD' + (AD + BD + CD). \end{aligned}$$

This means that

$$(AD' + BD' + CD') - (AD + BD + CD) \leq 3DD'.$$

In fact, you could make the difference on the left of this equation as small as you like by making D close enough to D' .

Since we know that $3DD'$ is a small quantity, this tells us that moving D to D' did not greatly affect the sum of the distances.

Problem 8 (Student page 90) One way to approach this problem is to draw a contour plot and see whether the contour lines, as they get smaller and smaller, converge to a line or a point. In this case, the contour lines seem to converge to a point.

Problems 9–10 (Student page 92) The length of the string represents (double) the distance to the cities. In this model, if the ring (airport) is at a point where there is any slack in the string, this means that “extra string” is being used, which translates to extra roadways being built to connect the cities to the airport. By pulling out the slack, you’re using the shortest bit of string possible, hence the smallest lengths of roads. The ring will then be at the best location for the airport.

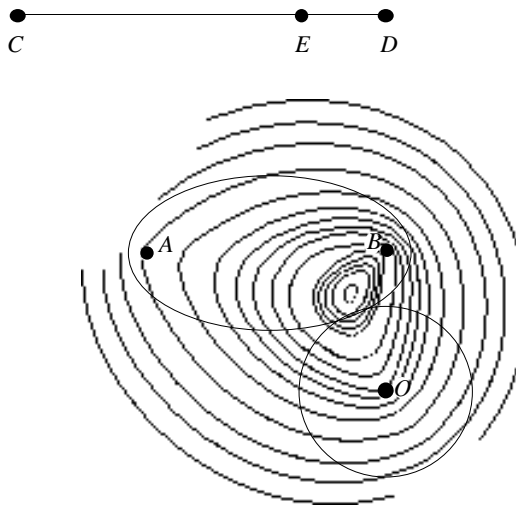
Problem 11 (Student page 92) The fact that this function (the function that takes any point and produces the sum of the distances from that point to the vertices of the triangle) has a unique minimum is not an easy thing to establish rigorously. You might think about questions like:

- If you laced the string differently (for example, starting at A rather than C) would the “best spot” come out the same?
- How do you know that, if the string is tight, the ring won’t move? (Try it with two cities; the string gets tight but the ring slides back and forth.)
- How do you know that, if the ring moves, the string is slack? (This is the same question as the previous one. Why?)
- What do the contour lines for this function look like?

Problem 12 (Student page 92) To use the string and nails device to draw contour lines, you need to allow a little slack into the system. Hold the string so that the amount of slack present remains constant as you move the ring around. By holding the amount of slack fixed, you will be tracing all points where the sum of the distances to the cities is some fixed value. This value is the length of the amount of slack you are allowing. By readjusting the amount of slack and starting over, you can draw a different contour line.

If you can put a pencil or pen in the center of the ring, you can draw a contour line.

In the solution for Problems 21–22 of Investigation 6.4, we discussed how to use Cabri Geometry II to draw contour lines for the function $P \mapsto PA + PB + PC$. These are contour lines for the airport function. Here's a picture from a Cabri session that shows some of the contour plot:



What is a *cam*?

Check that your string sketch produces “cam-shaped” contour lines like these. Notice how the contour lines tend to converge to one point. This point will be the best spot for the airport. Compare these contour lines with those for the following functions: (1) the function that minimizes total distance to two points, (2) the function that minimizes distance to the sides of an equilateral triangle, and (3) the function that minimizes distance to the sides of a nonequilateral triangle.

Problem 13 (Student page 92) Suppose that you use a contour plot to discover what seems to be the best spot for the airport, only to discover that there is some obstacle at this location, such as a power plant or a lake. Then you could choose the smallest contour line you can find which completely avoids this obstacle; any place on this line will be a good choice.

Problem 14 (Student page 93) If you experiment with the amount of slack you allow for each contour line, the construction in this problem will produce a family of ellipses. The connection between the contour lines for this problem, for the airport function, and for Rich’s function, suggests a whole family of functions that map points on the plane to the real numbers. See Problems 10 and 11 for more about generalized ellipses.

In this way, contour lines can be used to find second choice locations as well as the best spot.

For more information about this, see Cyril Isenberg's book *The Science of Soap Films and Soap Bubbles* (Dover Publications, Inc., New York, 1992).

Problem 15 (Student page 93)

- a. Liquid soap films have what students in one class called “a slight elastic quality”—they seek to cover minimal surfaces. Given a little time, the shape of an oblong bubble blown with a wand and set free to float away will tend toward a sphere. This has to do with surface tension and the general thermodynamic concept that systems always tend to minimize their overall energy.
- b.



The figure above shows the minimal road-building solution for an airport for four cities. Rather than a single point location, there is a line segment along which the airport might be built, and which represents some shared roadway between pairs of cities.

Problem 16 (Student page 94)

- a. The equilibrium point for this string-and-weights system solves the airport problem because when the weights hang freely, they pull out all the slack. This minimizes the amount of string above the board; as the string represents the sum of the distances from the knot to each of the three holes, it represents the solution to the airport problem.
- b. The four-hole system of strings will not show a minimal network of roads for the four-city airport problem if you tie the strings together at one spot. It does solve the problem for the case that none of the cities is able to share any roadway with another.

Computer Experiment (Student page 94) Minimizing the total distance means making the sum $j + k + m$ as small as possible. This corresponds to the point where P is closest to Q , so P will be at the endpoint of the locus it traces.

Problem 17 (Student page 95) If your software allows you to construct conic sections, intersect them and trace a locus; then you should be able to draw contour lines for the airport function. See the solutions for Problems 21–22 of Investigation 6.4 for more details.

If your software doesn't allow the above constructions, you can still create contour lines experimentally. Plot a point D and measure the sum $S = AD + BD + CD$. Then move D around while keeping track of the sum $AD + BD + CD$; plot a point every time you find a spot for D where this sum is this same value S . After you get enough points plotted, you will be able to see what the contour line for the value S should look like. You can then repeat this process for different values for the sum $AD + BD + CD$.

Problems 18–19 (Student page 95) These two problems are further opportunities to come up with your own conjecture for the airport problem. You may already have your conjecture prepared.

TESTING THE CONJECTURE

Problem 1 (*Student page 96*) Describe what you did that gave you insight on the problem, or describe what others did that made sense to you.

Problems 2–3 (*Student pages 97–98*) The gadget will always find a location for the airport, as long as no angle in the triangle formed by the cities is larger than 120° . (And as long as you don't mind extending the legs of your gadget whenever one or more of the cities is very far away from the others.) This type of gadget is really helpful in seeing that there should only be *one* best spot for the airport—it forces you to find that unique best spot in triangles where no angle is larger than 120° .

Problem 4 (*Student page 98*) This method does work. First, we will show that $\triangle ECB$ and $\triangle ACG$ are congruent:

We know that $\overline{EC} \cong \overline{AC}$ and $\overline{CG} \cong \overline{CB}$ because they are sides of equilateral triangles. Also, if we let $m\angle ACB = y$, then

$$m\angle ECB = m\angle ECA + m\angle ACB = 60^\circ + y,$$

and

$$m\angle ACG = m\angle ACB + m\angle BCG = y + 60^\circ.$$

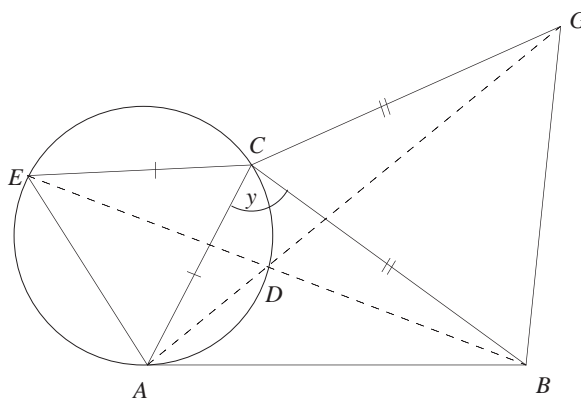
Therefore,

$$m\angle ECB = m\angle ACG,$$

and

$$\triangle ECB \cong \triangle ACG$$

by the SAS Postulate.



Now we know that

$$m\angle BEC = m\angle GAC$$

by CPCTC. Call this measure x . Next look at the circle passing through E , C , and D . It is clear that $\angle CAD$ and $\angle CED$ are the same angles as the two angles above, and so each has measure x . Since E lies on the circle, this means that A must also lie on the circle, for they are both angles of the same measure inscribed in the same arc.

So now we know that points E , C , D , and A all lie on this circle. Since $m\angle CEA = 60^\circ$, it follows that

$$m\angle CDA = 120^\circ,$$

as we would like. (Notice that $\angle CDA$ is an inscribed angle that intercepts a 240° arc.) Similarly we can show that

$$m\angle CDB = 120^\circ,$$

and then it has to follow that

$$m\angle ADB = 120^\circ.$$

Therefore, point D is the 120° point.

Problem 5 (Student page 98) Here is the complete solution to the airport problem:

If each of the angles of the triangle measures less than 120° , the best spot for the airport is at a point inside the triangle that makes 120° angles with the vertices. Otherwise, the best spot is at the vertex of the largest angle.

It's not so easy to verify every part of this result. The complete solution is outlined in the solution to Problem 13 of this investigation. Here is the part that you will eventually prove:

Such a point exists if and only if each of the angles of the triangle measures less than 120° . That requires yet another proof.

If there is a point inside the triangle that makes 120° angles with the vertices, it minimizes the airport function.

Problem 7 (Student page 99) For any triangle, the fairness solution will be the point which is equidistant from all three cities. This point is called the circumcenter of the triangle; it lies at the center of the circle circumscribed about the triangle. It is the spot where the perpendicular bisectors of the sides of the triangle intersect.

See the solution for Problem 6 of Investigation 6.21 for a detailed description of how to find this point.

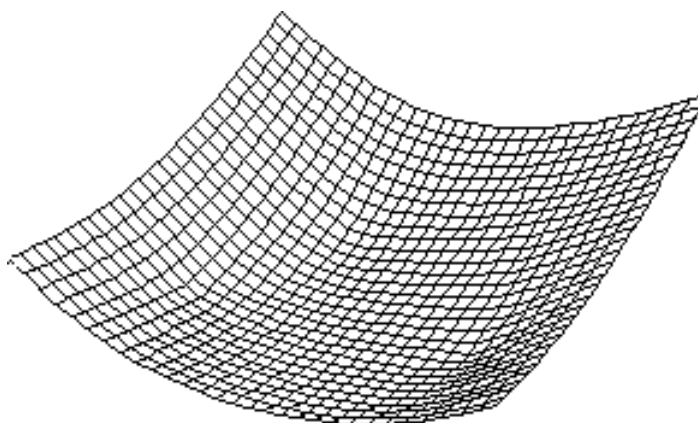
Something similar to the string-and-nails airport construction would help to draw these four-foci ellipses.

Problem 8 (Student page 99) By definition, an ellipse is the set of all points the sum of whose distances from two fixed points is constant. A contour line for the airport function is the locus of all points the sum of whose distances from *three* fixed points is constant. This is why you could say the contour lines are generalized ellipses; the only difference is the number of fixed points. An “ellipse” with four foci would be defined similarly, with the difference being that there are now four fixed points instead of two or three.

Problem 9 (Student page 99) One method of attack for this problem is to look at a contour plot for the three-city airport problem. If the best spot happens to lie on the line, you’re all set. If not, find the point on the line which is also on the contour line representing the smallest value.

Geometry software could also be used to find an experimental solution. Let D be any point on the line, and measure the sum $AD + BD + CD$; then move D along the line looking for the spot where the sum is smallest.

Problem 10 (Student page 99) A surface plot for the airport function looks like this:



The little bubbles in the middle are real; they’re anchored at points above the triangle’s vertices.

ESTABLISHING THE CONJECTURE

Problem 1 (*Student page 107*) If A' , D' , D , and C are all collinear, then $m\angle CDD' = 180^\circ$. Since $m\angle BDD' = 60^\circ$, this means that $m\angle CDB = 120^\circ$. This is exactly what we want.

Problem 2 (*Student page 107*) Yes, $m\angle ADB = 120^\circ$. Notice that $\triangle A'BD' \cong \triangle ADB$ by SSS: $BA = BA'$, $BD = BD'$, and $DA = D'A'$, all because of the rotation. Thus,

$$m\angle ADB = m\angle BD'A'.$$

We know that

$$m\angle BD'A' = 120^\circ,$$

since $m\angle A'D'D = 180^\circ$ and $m\angle DD'B = 60^\circ$, so

$$m\angle ADB = 120^\circ.$$

Also,

$$m\angle ADC = 120^\circ$$

since

$$m\angle BDA + m\angle ADC + m\angle CDB = 360^\circ.$$

Problem 3 (*Student page 107*) This is an important question. In order for the Hoffman proof to be valid, it has to work no matter which side of the triangle we use for the constructions. This means that, if we pick another side, we should get the same point. If not, something is wrong with the proof.

Problem 4 (*Student page 107*) The phrase “by symmetry” in the proof in the Student Module may seem confusing. What this means is that, if you start on the proof and work through it successfully to reach the statement that $m\angle BDC = 120^\circ$, you can then start all over again with different sides to obtain the same point for D and to prove that $m\angle ADB = 120^\circ$, and then a third time to prove that $m\angle ADC = 120^\circ$.

Problem 5 (*Student page 108*) An advantage is that readers will learn a great deal by struggling to fill in some of the details on their own. A disadvantage is that some readers may not be able to fill in the missing information or that what is written will be hard to decipher.

Problem 8 (*Student page 108*) If there is a point inside the triangle which forms 120° angles with each pair of vertices, it minimizes the airport function. If no such point exists, the minimum value can be found at the city that forms the largest vertex angle with the others.

- Start with triangle $\triangle ABC$ and a point D anywhere in the interior.
- Rotate $\triangle BDA$ by 60° counterclockwise about B , forming a new triangle, $\triangle BD'A'$.
- Move D around until you have found the location which makes the points A' , D' , D , and C collinear. This will be the desired location for D .

Problem 12 (*Student page 109*) If the third vertex is used for the Hoffman construction, the line segment connecting the outer vertex of the rotated triangle to the opposite vertex of the original triangle will pass through the point already found.

Problem 13 (Student page 109)

- a. D would be a silly location because the airport shouldn't be outside the triangle formed by the three cities—that would certainly require extra driving. You could prove this with the Triangle Inequality.
- b. The best spot here is the vertex of the largest angle.
- c. The Fermat point will be outside the triangle if any of the angles of the triangle measures more than 120° .

Here is what you've proven so far:

If there is a point inside the triangle that makes 120° angles with the vertices, it minimizes the airport function.

When does such a point exist? If all the angles of the triangle are less than 120° , the construction we gave in Hoffman's proof will, in fact, produce such a point. But (you can prove this as an exercise), if any angle of the triangle is bigger than 120° , the proof won't work. In this case, the student construction described in Problem 4 of Investigation 6.14 will produce a point outside the triangle that makes one 120° angle and two 60° angles with the vertices.

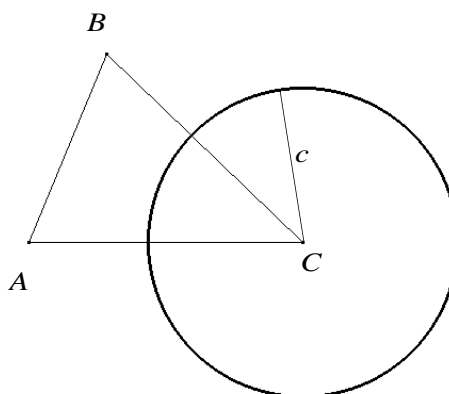
Let's back up and be a little more careful. The string experiment (see Investigation 6.13 of the Student Module) shows that there *is* a best spot to put the airport. Let's call that point D , and let $DA = a$, $DB = b$, and $DC = c$. There are two cases:

1. D is at a vertex;
2. D is *not* at a vertex.

Suppose D is at a vertex. Then a simple case analysis shows that D has to be at the vertex of the largest angle (because it is surrounded there by the two shortest sides of the triangle). That was easy!

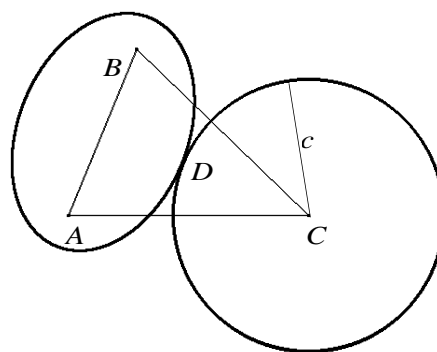
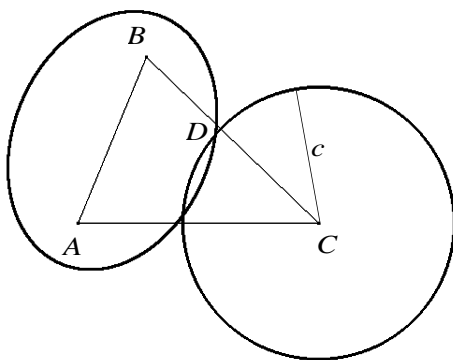
To really show that there is a best spot, you need ideas from calculus. The airport function clearly grows in value as you drag the point off to infinity, so you can confine attention to a large disc. But a theorem from calculus says that a continuous function on a disc actually attains a minimum value. Showing that there is only one in this case takes more work.

Now, suppose that D is not at a vertex. Then each of the distances a , b , and c is positive; D lies at the intersection of three circles, of radii a , b , and c , with centers A , B , and C respectively. Concentrate on C for a minute:



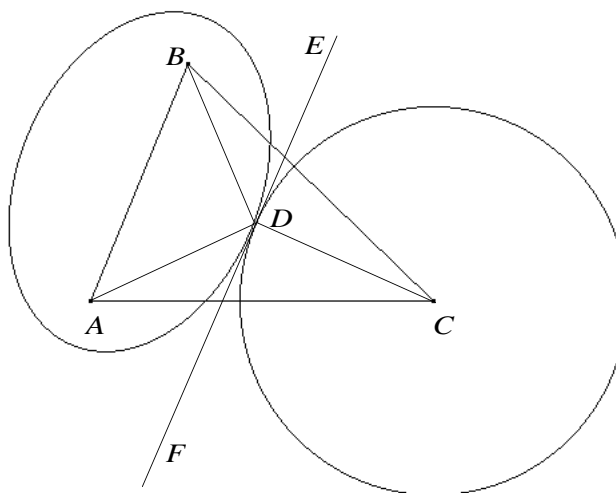
This makes nice connections with Problem 32 of Investigation 6.4 in the Student Module.

D must lie on a circle of radius c around C . Now, what point D on this circle minimizes $DA + DB$? Does this sound familiar? This is just like Problem 36 of Investigation 6.4 except this time A and B are *outside* the “circular pool.” We’ll prove they’re outside in a minute; let’s just assume for now that A and B are outside the circle. Which of these points D on the circle would you prefer?



In each picture, we’ve drawn the ellipse made up of points that have the same “sum of distances” from A and B as D does. In the first picture, there are points on the circle that are inside the ellipse, so, if we picked D at one of these points, we’d get a smaller

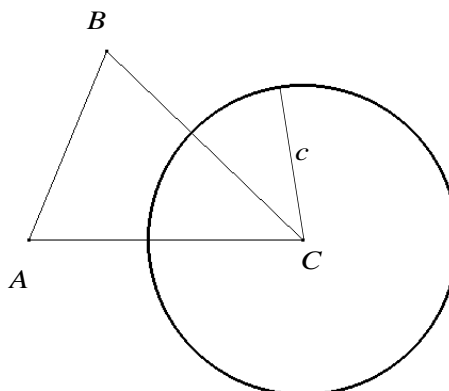
total sum. The conclusion is that the best spot for D is at the place where the ellipse with foci A and B is tangent to our circle. So, here's the situation:



Saying that “the ellipse with foci A and B is tangent to our circle at D ” just means that the ellipse and the circle share a common tangent line at D . By Problem 32 of Investigation 6.4, $\angle EDB \cong \angle FDA$. So, by addition, $\angle CDB \cong \angle CDA$. Now repeat the exact same construction with a circle of radius a around A concluding that, if D is the best place for the airport, $\angle ADB \cong \angle ADC$. So, if D isn't one of the vertices, it has to make three congruent angles (and hence, 120° angles) with A , B , and C .

This provides yet another proof that the Fermat point minimizes the sum of the distances.

Now for that unfinished piece of business: How do we know that, if D isn't at a vertex, then A and B are outside the circle?



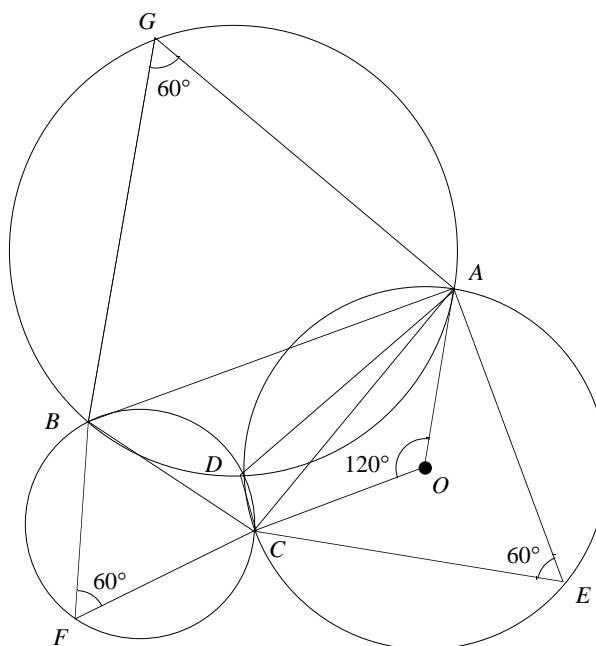
Suppose that A is inside the circle. Then $c > AC$. And, since D isn't a vertex, $AD + DB \geq AB$ (the Triangle Inequality). That is, $a + b > AB$. So, adding, we have

$$a + b + c \geq AC + AB,$$

and that says that A is at least as good a spot for the airport as D , so we could have taken D at a vertex, contradicting our assumption that the minimum wasn't at a vertex.

This gives a complete solution.

Problem 14 (Student page 110) Label the vertices of the new triangle as E , F , and G , as in the picture below.



Let O be the center of the circle circumscribed about $\triangle ACE$. Then since $m\angle CEA = 60^\circ$, we know that $m\angle AOC = 120^\circ$. So, for any point D on the upper arc between A and C , it follows that $m\angle ADC = 120^\circ$. We can apply similar reasoning to the circle circumscribed about $\triangle BCF$. Take D to be the point of intersection of these two circles. We can conclude that

$$m\angle ADC = m\angle CDB = 120^\circ.$$

This implies that

$$m\angle BDA = 120^\circ,$$

so D must lie on the third circle, also. Therefore, the intersection of the three circles is the Fermat point.

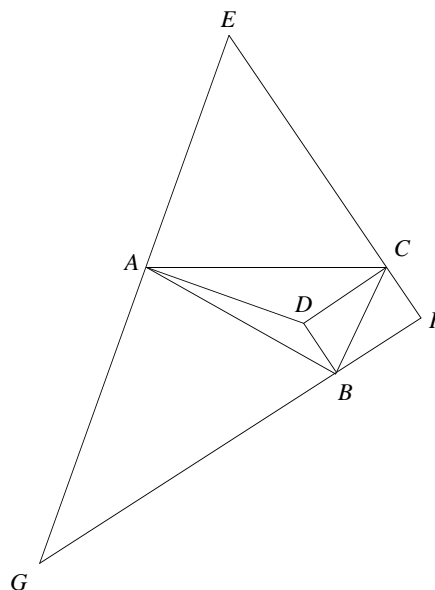
Problem 15 (*Student page 110*) Toricelli probably had already come up with the 120° conjecture before he began searching for a construction—a method to guarantee the exact location of that 120° spot. Maybe he thought of 120° as one third of a 360° circle, which inspired him to look at circles. Or maybe he was thinking he needed the intersection point of a couple of arcs of 120° , which he could do with equilateral triangles inscribed in circles because an inscribed angle is half the measure of the arc it intercepts. Do you have any other ideas?

THE AIRPORT REVISITED

Problem 1 (*Student page 111*) Both problems are examples of continuous systems, and both deal with a sum of distances to the triangle as a point moves throughout the interior of a triangle. When working on a problem, mathematicians often try to think of similar problems they have seen in the past and look for any connections that could help them find a solution. This is an important habit of mind.

Problem 2 (*Student page 112*) It is possible to make $\triangle EFG$ equilateral only when D is at the Fermat point. This will be true whether or not $\triangle ABC$ has an angle greater than 120° .

Cities A , B , and C are fixed. Pick a point D in the interior of $\triangle ABC$ and construct segments \overline{AD} , \overline{BD} , and \overline{CD} . Construct perpendiculars to these segments to form triangle $\triangle EFG$. As point D moves around, the segments \overline{AD} , \overline{BD} , and \overline{CD} change, as does $\triangle EFG$.



Now suppose D is at the Fermat point. This means that

$$m\angle ADC = m\angle ADB = m\angle BDC = 120^\circ.$$

By the construction, we also know that

$$m\angle DAE = m\angle DBG = m\angle DCE = 90^\circ.$$

Look at the quadrilateral formed by the points A , E , C , and D . The sum of the measures of the angles in any quadrilateral is 360° , so

$$m\angle DAE + m\angle AEC + m\angle ECD + m\angle CDA = 360^\circ$$

$$90^\circ + m\angle AEC + 90^\circ + 120^\circ = 360^\circ$$

$$m\angle AEC = 60^\circ.$$

We can repeat this process to establish that the measure of each angle in $\triangle EFG$ is 60° , so this triangle is equilateral.

Problem 3 (Student page 112) Suppose that $\triangle EFG$ is equilateral. D might be any point inside $\triangle ABC$, but because $\triangle EFG$ is equilateral,

$$m\angle G = m\angle E = m\angle F = 60^\circ.$$

As before, our construction guarantees that

$$m\angle DAE = m\angle DBG = m\angle DCE = 90^\circ.$$

Again look at the angles in quadrilateral $AECD$.

$$m\angle DAE + m\angle AEC + m\angle ECD + m\angle CDA = 360^\circ$$

$$90^\circ + 60^\circ + 90^\circ + m\angle CDA = 360^\circ$$

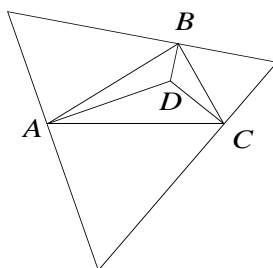
$$m\angle CDA = 120^\circ$$

We can repeat this process to establish that $m\angle ADB = 120^\circ$ and $m\angle CDB = 120^\circ$, so D is the Fermat point for $\triangle EFG$. Combining our work from Problems 2 and 3, we have shown that point D is the Fermat point if and only if the corresponding triangle $\triangle EFG$ is equilateral.

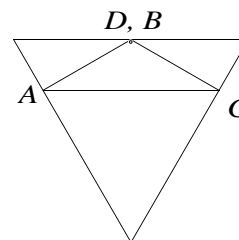
See the solution for
Problem 8 of
Investigation 6.7.

Problem 4 (Student page 113) Add perpendiculars from W to the sides of $\triangle EFG$. Use the fact that the sum of the lengths of the perpendiculars from D equals the sum of the lengths of the perpendiculars from W ; then use the Triangle Inequality to show that the perpendiculars from W are each less than each of the lengths WA , WB , and WC .

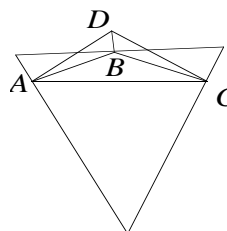
Problem 5 (Student page 114) These pictures show what happens to the equilateral triangle formed by constructing perpendiculars at the feet of the airport roads from the Fermat point as the obtuse angle of the inner triangle gets larger and larger:



$$m\angle B < 120^\circ$$



$$m\angle B = 120^\circ$$



$$m\angle B > 120^\circ$$

Here's an idea: Look at the Fermat point as a function of B . That is, if D is defined by the Toricelli construction, what happens to it as you drag B around?

The outer triangle is always equilateral, but as $\angle B$ grows, D passes outside the triangle and no longer makes three 120° angles with the vertices. In fact, it makes two 60° angles ($\angle ADB$ and $\angle CDB$ in this picture), or one 120° angle ($\angle ADC$). So, the construction still gives *something*, but the argument that it gives a *minimum* point doesn't work anymore. The point D is no longer a minimum—the minimum moves to B . For a complete proof that the minimum “sticks” to B , see the solution to Problem 13 of Investigation 6.15.

Problem 6 (Student page 114) Using what we've learned in this section, we know that the sum of the distances from the cities to the Fermat point will be equal to the sum of the distances from the airport (the Fermat point) to the sides of equilateral triangle EFG , which can be found by drawing perpendiculars to the airport roads. Since this sum is equal to the length of the altitude of $\triangle EFG$, it may be easier to find by measuring only the length of the side of the equilateral (and doing a few calculations) instead of trying to measure each of the three airport roads.

PROJECTS RELATED TO THE AIRPORT PROBLEM

Problems 1–2 (*Student page 117*) With four cities, you really need to start from scratch in the sense that you have to redefine what requirements the best location should satisfy. Geometry software is always nicely suited for studying problems involving continuous systems such as this. Many of the mechanical models can be useful here. Set up a system with four nails and string, or one with a board and four weights to help find the best spot if you're interested in minimizing the sum of the distances from the airport to the cities. The soap bubble model will be useful if you want to minimize the total length of new roadway to be built.

USING IDEAS FROM THE AIRPORT PROBLEM

Problem 1 (Student page 119)

- a. When thinking about what makes a good proof, there are two points of view to keep in mind: To a professional mathematician, a good proof might be one which is short, clean, and to the point—not encumbered with lots of unnecessary details. On the other hand, to a student, a good proof is one which is not extremely difficult to work through; this probably means one with lots of clear explanations. In either case, some things which may make a proof bad are poor grammar, awkward notation, explanations that are out of order, redundancy, and the inclusion of some fact or theorem which really isn't necessary to the proof.
- b. If each step in a proof can't be verified using the given hypotheses, the proof is invalid.

Problem 2 (Student page 119) There's no need here to provide a very technical definition of a rotation, but exactly what is being rotated, what point it is being rotated about, and what the angle of rotation is should be specified.

Problem 3 (Student page 119)

- a. $\triangle LPM$ has been rotated about the center, M , to $\triangle L'P'M$.
- b. We know that $m\angle LML' = m\angle PMP' = 60^\circ$ from the rotation. Also, since $MP' = MP$ and $m\angle PMP' = 60^\circ$, we know that $\triangle P'PM$ is equilateral. Thus

$$m\angle MP'P = m\angle P'PM = 60^\circ.$$

- c. All of the statements except the last are true:
 - i. $LP = L'P'$. This is true because the rotation doesn't change the lengths of the sides of the triangle.
 - ii. $MP = MP'$. This is true for the same reason.
 - iii. $\triangle MPP'$ is isosceles. This is true. In fact this triangle is also equilateral, from above.
 - iv. $m\angle MPP' = m\angle MP'P = 60^\circ$. This is also true; see above.
 - v. $MP + PN = PP' + PN$. This is true, since $MP = PP'$.
 - vi. $L'P' + PP' + PN = L'N$. This is false, unless L' , P' , P , and N are all collinear.

Problem 4 (Student page 120)

a.–c. $\triangle ABD$ has been rotated 45° about B to $\triangle A'BD'$.

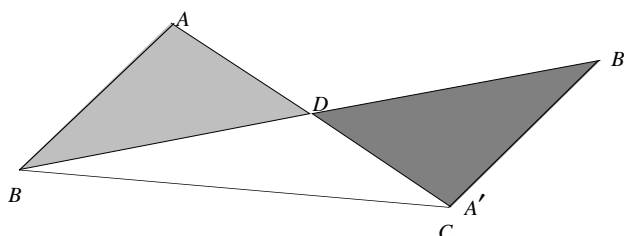
d. Here are nine pairs of congruent objects:

1. $\angle DBD' \cong \angle ABA'$ (Both angles measure 45° .)
2. $\overline{BA} \cong \overline{BA'}$
3. $\overline{BD} \cong \overline{BD'}$
4. $\overline{A'D'} \cong \overline{AD}$
5. $\overline{A'D'} \cong \overline{DC}$
6. $\triangle BA'D' \cong \triangle BAD$
7. $\angle A'D'B \cong \angle ADB$
8. $\angle ABD \cong \angle A'BD'$
9. $\angle DAB \cong \angle D'A'B$.

e. Using the given angle measures, we can determine the following angle measures:

- $m\angle BAD = 95^\circ$ since the sum of the measures of the angles in $\triangle ABD$ is 180° .
- $m\angle ABD' = 10^\circ$ since we know that $m\angle DBD' = 45^\circ$.
- $m\angle D'BA' = 35^\circ$ since $\triangle BA'D' \cong \triangle BAD$ (see part d).
- $m\angle CDB = 130^\circ$ since A , D , and C are collinear and $m\angle ADB = 50^\circ$.
- $m\angle DCB = 30^\circ$, since the sum of the measures of the angles in $\triangle DBC$ is 180° .
- Since $m\angle ADB = 50^\circ$, and $\angle ADB \cong \angle A'D'B$, we know that $m\angle A'D'B = 50^\circ$.
- $m\angle BA'D' = 95^\circ$ since $\triangle BA'D' \cong \triangle BAD$.

- f. If D were the center of rotation, and $\triangle ABD$ were rotated 180° , we would have the following picture:



Problem 5 (Student page 121) There is no rotation in this picture. What has happened is that $\triangle ABD$ has been reflected about \overline{AD} to form $\triangle ADB'$.

Problem 6 (Student page 121)

- a. A point of concurrency is a point where lines or line segments intersect. Here are definitions of the various points of concurrency in triangles:
- The *centroid* is the point where the medians are concurrent. It is also the center of gravity of the triangle.
 - The *circumcenter* is the center of the circle circumscribed about the triangle. It is the intersection point of the perpendicular bisectors from each side.
 - The *incenter* is the center of the circle inscribed in the triangle. It is the intersection point of the angle bisectors.
 - The *orthocenter* is the point at which the lines containing the altitudes of the triangle are concurrent. This point may fall outside of the triangle.
 - The Fermat point, for triangles with no angle measure greater than 120° , is the point making angles of 120° with each pair of vertices. It is the solution to the airport problem—it minimizes total distance to each of the vertices. For “very obtuse” triangles, the Fermat point sits at the vertex of the largest angle.
- b. If you wanted the airport to be the same distance from all three cities, you would find the circumcenter. If you were looking for a location which would be equidistant from already-existing highways that connect each one of the cities to the other two, you would be looking for the incenter. This is the point where the distances to all three sides of the triangle are equal.

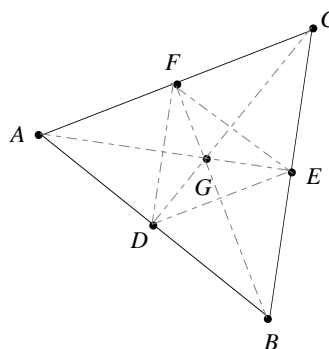
- c. Suppose you are asked the following:

At what point should a triangular board having uniform thickness and density be supported so that it will balance?

The answer is the centroid, as it is the center of gravity; it's also the answer to the question of where to put the airport when you consider population density. This site might be chosen if the goal is to cut down on air pollution created by the cars traveling to the airport.

Problem 7 (Student page 122)

- a. In an equilateral triangle, the medians, altitudes, angle bisectors, and perpendicular bisectors are all the same. Thus, G is also the orthocenter, the incenter, and the circumcenter. It is also the Fermat point. To see this, look at $\triangle AGB$: We know that $\overline{GD} \perp \overline{AB}$, and we also know that $m\angle GBD = m\angle GAD = 30^\circ$. By adding the angle measures in the two smaller triangles, $\triangle AGD$ and $\triangle BGD$, we see that $m\angle AGD = m\angle BGD = 60^\circ$. Therefore, $m\angle AGB = 120^\circ$. Repeating this argument for one other triangle, such as $\triangle AGC$ gives us that G is the Fermat point.



- b. We found the following:
- Segments \overline{CD} , \overline{BF} , and \overline{AE} are all medians. There are three more medians of small triangles, all the same length as \overline{GD} . Each median is also an angle bisector.
 - Triangle $\triangle EDF$ is equilateral, as each of its sides is a midsegment of $\triangle ABC$. Other equilateral triangles are $\triangle ADF$, $\triangle DBE$, and $\triangle ECF$.

- iii. There are six nonequilateral isosceles triangles: three similar to $\triangle AGB$ and three similar to $\triangle GDE$.
- iv. The following quadrilaterals are trapezoids: $FEBA$, $DFCB$, and $EDAC$.
- v. The following quadrilaterals are rhombi: $EDAF$, $DFCE$, and $EFDB$.
- vi. Kites: $ADGF$, $BDGE$, $CEGF$, three formed with the vertices F , D , G , and E , and three formed with the vertices and center of the large triangle, for example $GACB$.
- vii. We didn't find any other convex four-sided polygons.
- viii. There's not enough asymmetry to allow for any other convex four-sided polygons besides the kites listed above.
- ix. Concave five-sided polygons: three congruent to $ABEGF$ and six congruent to $ADGEF$.
- x. The equilateral triangles mentioned earlier are all regular polygons.
- xi. There are six right congruent triangles, of which $\triangle ACD$ is one.

Problem 8 (Student page 123) C must be the point $(2, 2\sqrt{3})$.

- a. The slope of \overleftrightarrow{AC} is $\sqrt{3}$.
- b. Point D is on \overleftrightarrow{AB} , so the slope is 0.
- c. An equation for \overleftrightarrow{BC} is $y = -\sqrt{3}x + 4\sqrt{3}$. An equation for \overleftrightarrow{AF} is $y = \sqrt{3}x$. (These equations are written in slope-intercept form; they can also be written in other forms.)
- d. $C = (2, 2\sqrt{3})$, $D = (2, 0)$, $G = (2, \frac{2\sqrt{3}}{3})$.
- e. The length of \overline{AF} is 2.

Problem 9 (Student page 123) The essential idea of the Hoffman proof is to turn the problem into one where the thing you want to minimize is a path between two points. Then all you have to do is to draw a straight line. That's exactly the key behind the run-and-swim problem, the burning tent problem, and other "shortest path" problems.

THE ISOPERIMETRIC PROBLEM

Problem 1 (*Student page 124*) For a polygon with any number of sides, the solution to the isoperimetric problem will be the one which is *regular* (all sides and all angles congruent). For triangles, this means an equilateral triangle.

Here's an application of the isoperimetric problem for triangles:

An architect for a museum wants to construct a triangular reflecting pool with a marble bench around it to place in front of the building. She can only afford enough marble to build 45 feet of benches. What will be the dimensions of the largest such pool?

The rest of this section of the module will focus on *the* isoperimetric problem—the one that considers all closed curves of a fixed perimeter—as well as a proof, some history, and various extensions of the problem.

Problem 2 (*Student page 124*) A circle is the closed curve which encloses the greatest area. One way to see this is by using a piece of string with the ends tied together. All shapes you form with the string will have the same perimeter, since the length of the string is never changed. The area enclosed by the string will not be the same, however. If the string forms a square, for example, you can create more area simply by “rounding out” the sides of the shape. A circle made of string cannot be adjusted to create more enclosed area.

Here's another way to visualize what's happening: Picture a closed loop of rope lying on a flat surface in no particular shape. Pretend that some liquid is poured into the center of the rope, and the liquid is pushing the rope out as far as it can be pushed. Once the rope is pushed into the shape of a circle, it can't be pushed outward any farther.

Problem 3 (*Student page 124*) Here is one idea that can be used to try to prove the conjecture: If you take a closed curve with a given perimeter that is *not* a circle, show that you can always find a different curve that has the same perimeter yet encloses more area.

Another approach is to make a list of properties belonging to the curve that encloses the most area. Then perhaps you could show that this set of properties is unique to circles. For example, suppose you could show that the best curve consists of all points equidistant from some fixed point; this would let you conclude that the best curve is a circle.

The term *closed curve* includes curves made up of line segments as well as of arcs. Rectangles, crescent moons, and loops that cross themselves can all be closed curves.

This idea will come up again in later problems in this investigation, for example, Problems 13 and 14.

We will always use the term *best curve* to mean the closed curve which encloses more area than any other closed curve with the same perimeter.

For examples of optimization problems with either no solutions or more than one solution, see Problems 6, 7, and 15 in this investigation.

There are always new mathematical proofs being produced—mathematics is far from a finished subject.

Problem 4 (Student page 124) Suppose you had two different (noncongruent) closed curves, A and B , with the same perimeter. Suppose also that the area inside A equals the area inside B . Now if *every other* closed curve with that same perimeter encloses *less* area, then both A and B could be considered best curves. This will not turn out to be a possibility, but at this point there is no reason to believe this couldn't happen.

Or, suppose that no matter what closed curve of a specific perimeter you choose, there is *always* another curve of the same perimeter with *more* area. This would mean there would be no best curve at all. As we'll soon see, this is not the case either.

Problem 5 (Student page 125) Proofs play a vital role in mathematics. Although outside of mathematics class many statements may be taken at face value, in the field of mathematics a statement must be proved before it can be accepted. For example, for centuries, most mathematicians believed Fermat's conjecture (known as Fermat's Last Theorem) to be true (see Investigation 6.15), although a proof had not been found. Numerical data supported it in an enormous number of cases. Yet many people worked on finding a proof, because, not until a proof was found, did the conjecture become a definitive mathematical theorem.

Think about how this emphasis on proof in mathematics is different from the nature of many other fields. Do you prove conjectures in a paper for English class, or does it suffice to provide facts which support opinions? What's the role of making conjectures in a math class?

For Discussion (Student page 126) The trickiest part here is seeing why Step 3 implies that you have a circle. See the solution to Problem 22 for a complete explanation of this question.

Problem 6 (Student page 127) Here's an example: Think of f as a function defined on the open interval $0 < x < 1$, given by $f(x) = x + 1$. What is the maximum value of f ? ("Open interval" in this case just means that x is "strictly less than" 1 and "strictly greater than" 0; it never takes on either of those two values.) The function will never have a maximum value, because you can keep taking x 's closer and closer to 1, yielding larger and larger values for $f(x)$. $f(x)$ gets closer and closer to 2, but doesn't get there because x can never actually equal 1.

Here's another example: With no restriction on the perimeter, draw a triangle with the largest possible area or a triangle with the smallest possible area. (Clearly, you can't; if it has zero area, then it's not really a triangle.) One reason the isoperimetric

problem has a solution, on the other hand, is that the perimeter of the curve is kept fixed.

Problem 7 (Student page 127) The problem “find the shortest path from A to B without touching the circle” is an optimization problem which has no solution because, no matter what path from A to B your partner draws, you can always draw yours so it’s closer to the tangent to the circle that’s parallel to \overline{AB} (without touching it). This will make your path shorter. In theory, no one can ever win this game, although, since our pencils aren’t infinitesimally narrow, someone might attempt to argue that they could!

Problem 8 (Student page 127)

- a. Here is one way to restate and summarize the proof:

We assumed that x was the largest number smaller than 1, and then using this we performed some calculations and discovered that x was greater than or equal to 1. This doesn’t make any sense, so such a number x doesn’t exist. Thus, there is no largest number less than 1.

This shows the thinking of someone who could look beyond the algebraic details and see the heart of the proof.

- b. The above summary misses one important point, however. The number $x + \frac{1-x}{2}$ is the number *precisely* halfway between x and 1 (the distance between x and 1 is $1 - x$; add one half of this distance to x to get the midpoint). Using this, the idea of what is happening in the proof can be expressed differently:

We assumed that x was the largest number less than 1. But we can always find a number which is exactly halfway between x and 1, namely $x + \frac{1-x}{2}$. This midpoint is less than 1 but greater than x (simply due to its position on the number line). Thus, we know there can’t be a *largest* number less than 1 because another number (the midpoint) can always be found which is *bigger* than the supposedly-largest number less than 1 and is still less than 1.

Realizing that $x + \frac{1-x}{2}$ is halfway between x and 1 is a little subtle, but you might boil it down to one student’s statement that “If there’s any difference between x and 1, you can always split the difference in half and have a number still larger than x .”

- c. In this particular proof, you assume that a largest number less than 1 exists, and call it x . This assumption then leads to a contradiction (you show that $x \geq 1$). Therefore the assumption must have been false, so there is no largest number less than 1.

This argument is an example of a *proof by contradiction* or *indirect proof*. It's a hard concept to get used to; you assume the exact opposite of what you want to prove and proceed to show that this gets you into trouble and yields a contradiction. By supposing the opposite and reaching a contradiction, it means that what you wanted to prove originally must be, in fact, true.

Problem 9 (Student page 129) Many optimization problems are solved by first *finding* the best point (or number, position, curve, and so on) experimentally and then proving it is the optimal one. The isoperimetric problem is perhaps a little more mysterious in that the defining properties of the best curve are not known right away, and the proposed “detective method” will be used to deduce them by first assuming that a unique best curve exists.

Problem 10 (Student page 130) This is a classic problem, and there is a lot happening here. The important idea to grasp is that letting x be the sum is not a valid thing to do. We define x as an infinite sum, but how do we know that this is a number which even *exists*? Actually, x is defined by way of a *geometric series*—an infinite sum of positive powers of some number called the *ratio*. Here the ratio is 2, as x is defined as an infinite sum of powers of 2.

A geometric series is said to *converge* if the sequence of partial sums (say, the sum of the first 10 terms, the sum of the first 100 terms, and so on) gets closer and closer to a specific number. More precisely, the series converges to L if you can get as close as you like to L by adding up enough terms of the series. The series *diverges* if, as you add more and more terms, the partial sums get larger and larger (in absolute value). Moreover, a geometric series converges only if the absolute value of the ratio is less than 1. Hence, in this example it diverges; x is not a finite number you can work with, and all the remaining calculations in the Student Module for this problem are invalid. Notice that solving the equation $x = 2(1 + x)$ gives $x = -2$, which doesn't make sense in the original problem.

Problem 11 (Student page 130) Suppose the curve pictured in the Student Module is the best possible curve. This would mean that it encloses more area than any other curve with the same perimeter. But, we can increase the area by removing the concavity of the polygon, either by imagining hinges at the vertices or (more rigorously) by reflecting the “offending” vertex over the line connecting its two adjacent vertices.

There exists a number system in which the notion of absolute value is defined differently, causing the given geometric series x to actually converge. This system is called the *2-adic numbers*.

This argument shows that the best possible curve must be *simple*, that is, a curve which doesn't intersect itself.

Of course, simply applying a theorem from the *Student Module* isn't as much fun as discovering the answer yourself. You could experiment by comparing the area of several quadrilaterals of equal perimeter to understand the theorem for yourself. Examples of such arguments are included in the solution for Problem 3 of Investigation 6.21, and in the "Mathematics Connections" section of the *Teaching Notes* for this investigation.

Neither step affects the perimeter, but area is increased, contradicting the fact that this was supposed to be the best curve.

Problem 12 (*Student page 130*) Suppose you have a curve made out of a closed piece of string arranged like a figure eight. You could just "open up" the curve at the center of the eight to get a new shape that doesn't cross itself. Put another way, any time a figure crosses itself it will be a concave figure. We have shown that concavity in a figure is a property that can be removed to yield more area for the same perimeter. This new shape definitely encloses more area than the figure eight, but it has the same perimeter since the length of the string has not changed. Thus, you have a new curve with the same perimeter that encloses more area. Now you can generalize this to any curve which "crosses itself."

Problem 13 (*Student page 130*) Refer back to Theorem 6.2 in Investigation 6.3 of the *Student Module*. Here is that theorem again:

THEOREM

Of all the polygons with a given perimeter and a given number of sides, the regular polygon has the most area.

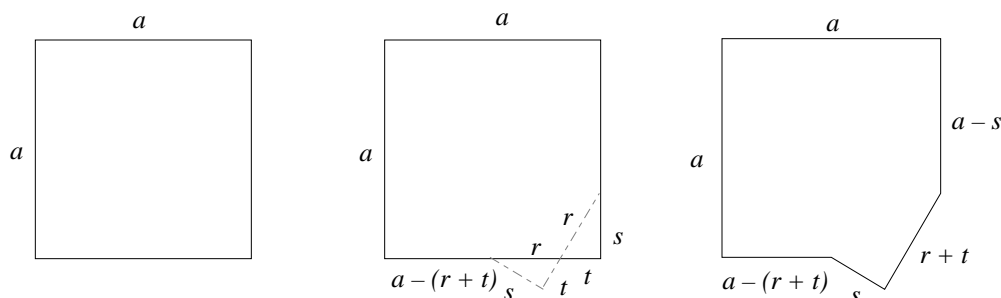
Using this theorem, if the best possible curve is a quadrilateral, it must be a square.

Problem 14 (*Student page 130*) In Problem 13, we concluded that if the best possible curve is a quadrilateral, then it must be a square. If we now show that a square cannot be the best curve, it will then follow that the best curve can't be a quadrilateral at all.

Suppose you have a square of sidelength a which is the best curve. This means that the square has more area than any other closed curve with perimeter $4a$. Now consider a circle with radius $r = \frac{2a}{\pi}$. This circle has circumference $2\pi r = 4a$, the same as the perimeter of the square. The area of the square is a^2 while the area of the circle is $\pi r^2 = \frac{\pi 4a^2}{\pi^2} = \frac{4a^2}{\pi}$. This is approximately equal to $1.3a^2$, which is greater than a^2 , the area of the square.

So, given a square, we are always able to find a circle with the same perimeter that encloses more area, which means that the best curve can't be a square.

Here is another argument that it doesn't require any area formulas. Consider a square of sidelength a . Cut off the lower right corner of the square, calling the lengths of the sides of the resulting triangle r , s , and t .



Now reflect the triangle about its lower left vertex, lining up the hypotenuse with the base of the square. You now have a hexagon with the same area and perimeter as the original square. The area is the same since you just cut off a piece of the square and glued it elsewhere. To see that the perimeter is the same, add up the lengths of the six sides. Starting at the top and proceeding clockwise, you get

$$a + (a - s) + (r + t) + s + [a - (r + t)] + a = 4a.$$

Let's call this new curve C . The important fact about C is that it has the same area and perimeter as the square, but is *concave*. This means (from Problem 13) that there exists a different curve with the same perimeter as C , but which encloses more area (remove the concavity by a reflection). So the best curve can't be a square!

This argument can be generalized to a polygon with any number of sides. If a polygon is assumed to be the best curve, cut off a corner and attach it elsewhere to obtain a concave polygon with the same perimeter. From this, you can remove the concavity to obtain another curve with more enclosed area. We can now conclude that the best curve cannot be any type of polygon.

Problem 15 (Student page 130) Suppose you're asked to find the shortest path between the North Pole and the South Pole (or any other two points exactly halfway around the Earth from each other). This optimization problem has more than one solution—a shortest path could go in any one of an infinite number of directions.

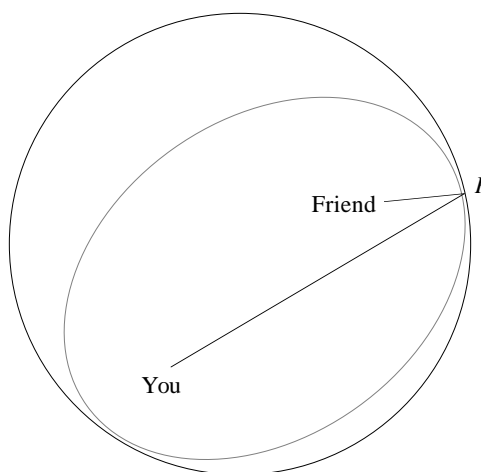
Other situations may involve some finite number of solutions. Say you're in a city where the streets are laid out in a grid. There will be more than one shortest path to

What else might be tangent to an ellipse in more than one place?

Two spots represent the shortest path from your position to the side of the circle to your friend's position.

drive from one point to another (or on to a third, and so on), as long as your travel involves two locations that aren't both on the same line. If you want to move three blocks east and one block north, for example, there are four shortest paths; if you want to move two blocks west and five blocks north, there are 21 shortest paths.

Or think about ellipses. All the points on an ellipse represent places of constant value for the function that gives the sum of the distances from each focus to those places. Any object that can be tangent to an ellipse in more than one place, such as a circle, a rectangle, or a square, will yield a finite number of minimal path solutions (from one focus to the object to the other focus).



Problem 16 (Student page 130) Recall that the detective method is being used here to find the best possible curve. We are assuming such a curve exists, and we want to discover its important characteristics. Each of Problems 13–16 can yield a little more insight as to what this curve must be.

The result in Problem 13 shows that, if the best curve is a polygon, it has no concavity. Such a curve is called *convex*. Then Problem 14 shows that the curve must be simple. Through Problem 15, you can show that if the best curve is a quadrilateral then it must be a square, but the conclusion of Problem 16 is that a square cannot be the best possible curve. Recall also that Problem 16 could be generalized to show that the best curve can't be any polygon at all. Therefore, at this point, the best curve is simple, has no concavity, and is not a polygon.

In the same way that we know that there is some line which divides the surface area of an irregular-shaped lake, we also know that there exists some line which divides the perimeter of any curve in half.

Refer back to ideas from the discussion question following Problem 9.

A convex, simple, nonpolygon: how did we find these characteristics? In each case, it was assumed that the curve had a specific property, and then a curve with the same perimeter was found which enclosed more area, showing that the original curve could not have been the best possible one. This is the main strategy being employed throughout this investigation.

Problem 17 (Student page 131) Here's an algorithm for constructing a line segment which bisects the perimeter of the curve. Recall that the best possible curve is simple, that is, it never crosses itself. Suppose the perimeter of the curve is x . Choose a point P on the curve; starting at P , move along the curve in either direction a length of $\frac{1}{2}x$. Call the point where you stop Q . Line segment \overline{PQ} will bisect the perimeter.

Problem 19 (Student page 132) The existence hypothesis is crucial in solving the isoperimetric problem. You can't proceed with Steps 2 and 3 unless you've established, or assumed, that a best curve exists.

Problem 20 (Student page 132) We've established that the isoperimetric curve must be a simple, closed, convex, nonpolygonal curve with the additional property that any line that bisects the perimeter must also bisect the area (or it couldn't qualify as a contender for the best curve). Ellipses, egg-shapes, and other simple closed curves made up of arcs still satisfy these properties. Not until Step 3 has been established will noncircles be ruled out.

Take a look at the various kinds of lines that can bisect both the perimeter and the area for these figures. Find some examples of lines that bisect perimeter and area for an ellipse, but which are not lines of symmetry. Can you find such a line for a symmetrical egg-shape? What about a nonsymmetrical egg-shape?

For Discussion (Student page 132) Suppose we have a closed curve which is the solution to the isoperimetric problem. Then the top portion of the curve must solve the reduced isoperimetric problem. For if it didn't, there would be a curve of the same length with endpoints on the diameter but enclosing more area. Then we could flip this curve about the diameter to obtain a closed curve with more area than our best closed curve, which isn't possible.

This technique—looking at half of the geometric figure for which you're looking to maximize area—is essentially the same as the one used for several problems in Investigation 6.3, most notably the pen-against-the-wall problems. For example, if trying to maximize area for a five-sided fenced-in pen when one of the sides “doesn't count” because it's a stone wall, it suffices to look at the four sides as forming half of a larger figure to be maximized. In other words, it suffices to look at the shape that maximizes

area for an eight-sided figure (the regular octagon). The largest five-sided pen, then, is half a regular octagon.

Similarly, if the circle is the shape that maximizes area for a given perimeter, then of all the curves that have the same perimeter and two endpoints on a line ℓ , the semicircle whose diameter lies along ℓ is the shape that maximizes area.

Problem 21 (*Student page 133*) In Problem 6 of Investigation 6.3, we wanted to maximize the area of a rectangular pen built against the wall. What we did was look at the union of the pen and its reflection image over the wall and saw that the area was maximized when we had a square, so we knew that the best pen would be half the square. But in that problem we already knew that the best figure would be a square, so we used it to solve the problem. If we already knew that the solution to the isoperimetric problem was a circle, then we could use the same technique to show that the solution to the reduced isoperimetric problem is a semicircle, but we don't yet know that as we haven't finished the proof.

Notice we are *not* saying that we can use the isoperimetric conjecture to solve the reduced isoperimetric conjecture or vice versa. We are simply saying that the two conjectures are *equivalent*, so instead of proving the isoperimetric conjecture it's OK if we prove the reduced conjecture. Be careful; this is tricky.

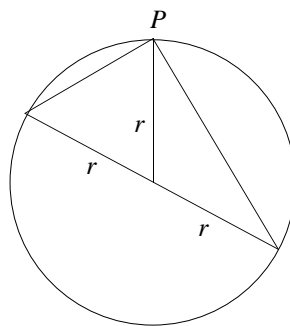
Problem 22 (*Student page 133*) Yes, if the angle formed by any point on the curve and the two endpoints of the diameter is a right angle, then the curve is guaranteed to be a semicircle. To see this, first suppose that, for any point P on a given closed curve, the line segments from P to the endpoints of the diameter form a right angle. We'll show the curve is a circle. Draw a third line segment, a median from P to the diameter, creating two triangles. Let r be the length of half the diameter. Now apply the following theorem:

THEOREM

The midpoint of the hypotenuse of a right triangle is equidistant from the vertices of the triangle.

This theorem shows that the distance from P to the midpoint of the diameter is also equal to r . Therefore, we know that if at every point P there is a right angle, then

every point P is the same distance from the midpoint of the diameter. Thus, the curve we are working with must be a circle.

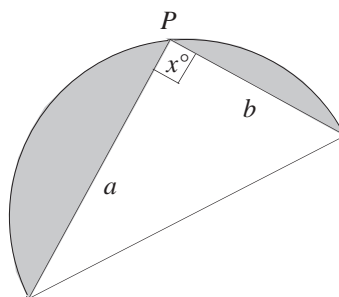


When we prove that the statement in Step 3 is true for the curve which encloses the most area, we will have proved that this curve is a circle. This is because the property in Step 3 is one which is intrinsic to circles. (An angle inscribed in a semicircle is a right angle.)

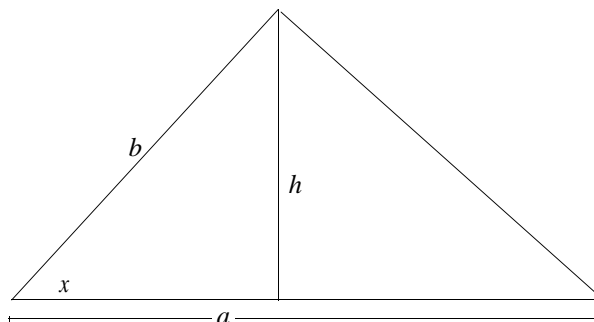
If you did Problem 1 in Investigation 6.3, you may remember an argument based on having two fixed sidelengths and the area formula for triangles. If not, see the solution for that problem.

Problem 23 (Student page 135) The idea here is that, if the angle is less than 90° , the “hinge” at P can be opened more, forming a larger angle and a curve enclosing more area. However, once the angle reaches 90° , widening it more will not help. For if the hinge is opened past 90° , the curve will be pushed downward, causing the amount of enclosed area to shrink. Therefore, the enclosed area is maximized when the angle is a right angle. The problem is presented more formally below.

Let P be the point on the curve, x the measure of the angle at P , and a and b the lengths of the line segments from P to the endpoints of the diameter.



The area under the curve will be at a maximum precisely when the area of the triangle is also maximized (because the shaded area is fixed). Let's look at the triangle separately, oriented so that its base has length a . Let h be the length of the altitude. We want the area of the triangle to be largest when x is 90° .



From the solution to Problem 1 of Investigation 6.3: “the area of the triangle will be half the product of one of the given sides and the height, and the height is at a maximum when it coincides with the other given side (that is, when the given sides a and b are the height and base of the triangle, and the included angle, therefore, is 90°).”

Notice that the area of the triangle is given by $\text{Area} = \frac{1}{2}ah$. As angle x changes, h will change, but we will always have $h \leq b$. When x is less than or greater than 90° , b is the length of the hypotenuse of a right triangle which has one side of length h , in which case $h < b$. When x is a right angle $h = b$. So $h \leq b$. Therefore, since the area $\frac{1}{2}ah$ is largest when h is largest (since a is fixed), we know it will be maximized when $h = b$ and x is 90° .

If you know a little trigonometry, you can see this more rigorously. Looking at the right triangle formed by the altitude, with hypotenuse of length b , notice that $\sin x = \frac{h}{b}$ implying $h = b \sin x$. So the area of the triangle is given by $\text{Area} = \frac{1}{2}ah = \frac{1}{2}ab \sin x$. Since a and b are fixed lengths, the area will be largest when $\sin(x)$ is largest, that is, when x is 90° .

Problem 24 (Student page 135) A complete proof will:

- State the problem.
- State the existence hypothesis.
- Eliminate curves which are concave or which are not simple.
- Show the best curve has a diameter which bisects the area.
- Explain the reduced isoperimetric conjecture.
- Include and explain the right angle argument.

- Explain why the right angles for every point P on the curve implies that a semi-circle is the solution to the reduced isoperimetric problem.
- Provide a conclusion.

A PROBLEM WITH A LONG HISTORY

Of course this depends a little on the sidelength as well: a regular polygon with 1000 sides of length $\frac{1}{4}$ cm will look very much like a circle, but what about sides of length 1 meter?

Problem 1 (*Student page 136*) Princess Dido used the fact that a semicircle encloses more area than any other curve with its endpoints on a fixed line; this is essentially the reduced isoperimetric problem.

Problem 2 (*Student page 136*) It is reasonably clear from the context in which this sentence appeared that we’re talking about the converse of the isoperimetric problem: given a fixed area, what shape has the smallest perimeter? The answer is the same: a circle. From a medical point of view, assuming that a wound heals first along and near its edge, a wound with a small perimeter would heal the least per day and therefore heal most slowly. Aristotle implies, without showing the details, that a geometer would not only know that a circle has the smallest perimeter for a given area but would also be able to prove it.

Problem 3 (*Student page 137*) Ptolemy said that one polygon is “more polygonal” than another if it has more sides. If you consider two regular polygons of the same perimeter, the one with the greater number of sides will enclose a larger area.

Ptolemy concluded that because adding more sides often increases the area, the circle must contain the most area. This is because he is viewing the circle as a polygon with infinitely many sides! Look at a series of regular polygons with a successively increasing number of sides, you’ll see that the shapes of the polygons begin to approach the shape of a circle. Thus, the circle can be thought of as the limiting shape for the regular polygons.

Problem 4 (*Student page 137*) Zenodorus’s first theorem appeared in Investigation 6.6. For example, we know that an equilateral triangle with sides of length 6 has more area than any other triangle with perimeter 18, and a square with sides of length 5 has more area than any other quadrilateral with perimeter 20.

Zenodorus’s second theorem says that a circle with the same perimeter of any regular polygon will have the greater area (see Problem 16 of Investigation 6.19 for the example of a square). Thus, suppose you have a circle of radius 3. Its circumference is 6π , and its area is 9π . This means that any other polygon of perimeter 6π will have area *smaller* than 9π .

In the third theorem, a segment of a circle refers to part of the interior of a circle—it is that piece which is bounded by both an arc and a chord of the circle. Zenodorus proved that, of all segments of circles with a fixed perimeter, the semicircle has the greatest area. For instance, suppose we consider all segments of circles with perimeter $5\pi + 10$. The one with the largest area would be a semicircle; in fact it would be half of

a circle with diameter 10. (The perimeter of a semicircle is one half the circumference plus the diameter, which in this case is $(\frac{1}{2})(\pi)(10) + 10 = 5\pi + 10$.)

Problem 5 (Student page 137) A segment of a circle looks like the area enclosed by a simple curve (the arc of the circle) with endpoints along a fixed line segment (the chord of the circle). Asking which segments of circles with fixed perimeter have largest area is really just asking the reduced isoperimetric question, which is exactly what Dido solved.

Problem 6 (Student page 137) Using Zenodorus's first theorem, we see that, given a fixed perimeter, if the best curve is a polygon then it must be a regular polygon. His second theorem then tells us that the circle having that fixed perimeter contains more area than any regular polygon with the same perimeter, no matter the number of sides. This doesn't tell us that the circle is the best curve; it only shows that the circle is better than any polygon.

Zenodorus did not prove the isoperimetric conjecture. For example, do his theorems convince us that a circle encloses more area than an ellipse with the same perimeter? Does he tell us a circle contains more area than a figure eight with the same perimeter? Zenodorus contributed a very important part of the solution, but not the entire proof.

Problem 8 (Student page 138) It's certainly easier to make the conjecture that of all curves with a fixed area, the circle has the smallest perimeter. You can compute lots of examples by hand or with software—for example, suppose we fix an area of 100. A square of area 100 would have to have sides of length 10 and perimeter 40. Consider a 25×4 rectangle. It too has area 100, and its perimeter is 58. A right triangle with base 4 and height 50 has area 100, and its perimeter is approximately 104. A circle of area 100 has a radius of $\frac{10}{\sqrt{\pi}}$, so its circumference would be $20\sqrt{\pi}$, which is approximately 35. This leads to the conjecture that the circle will have the smallest perimeter of all closed curves with area 100.

See Problem 24 of Investigation 6.3 for a similar proof.

The proof is trickier. First we present a proof for a specific area (area 100). This is followed by a general proof for any arbitrary fixed area.

Area 100: The circle with area 100 has radius $\frac{10}{\sqrt{\pi}}$ and circumference $20\sqrt{\pi}$. We want to show that *any other* shape with area 100 has perimeter greater than $20\sqrt{\pi}$. Suppose that S is a shape such that $\text{Area}(S) = 100$, and S has some perimeter k with $k \leq 20\sqrt{\pi}$. Look at the circle C with perimeter equal to k . If r is the radius of C , $2\pi r = k$, so $r = \frac{k}{2\pi}$. Now C and S have the same perimeter, but we know from the isoperimetric problem that the circle has more area. So $\text{Area}(C) > \text{Area}(S)$.

We know $\text{Area}(C) = \pi r^2 = \pi \left(\frac{k}{2\pi}\right)^2 = \frac{k^2}{4\pi}$. So,

$$\text{Area}(C) > \text{Area}(S)$$

$$\frac{k^2}{4\pi} > 100$$

$$k^2 > 400\pi$$

$$k > 20\sqrt{\pi}.$$

This is a contradiction since we assumed above that $k \leq 20\sqrt{\pi}$. This means there is no shape of area 100 with perimeter smaller than that of the circle of area 100.

General proof: Suppose we fix an area A . We want to show that the circle of area A has smaller perimeter than any other curve of area A . A circle of area A has radius $\sqrt{\frac{A}{\pi}}$ and perimeter $2\sqrt{A\pi}$. So suppose S is a closed curve with area A , but that the perimeter of S is k and $k \leq 2\sqrt{A\pi}$. Let C be the circle of perimeter k . So if C has radius r we know $r = \frac{k}{2\pi}$, and so $\text{Area}(C) = \frac{k^2}{4\pi}$. As before, we have that C and S both have perimeter k , but C has greater area since it's a circle. So,

$$\text{Area}(C) > \text{Area}(S)$$

$$\frac{k^2}{4\pi} > A$$

$$k^2 > 4\pi A$$

$$k > 2\sqrt{A\pi}.$$

Again we have a contradiction, so S can't exist. Thus, of all closed curves with area A , the circle has the smallest perimeter.

Below is another argument; it's perhaps more intuitive, but it's not as concise.

Fix an area A . Look at all closed curves enclosing this area. One will be the circle with area A ; call it C . Now imagine taking every other curve and either stretching or shrinking it so that it has the same perimeter as C . Now we have a collection of closed curves all with a fixed perimeter, although their areas vary. The isoperimetric problem says that C encloses more area than any of these new curves. In other words, all these curves have area less than A , so all of the original curves had to be shrunk in order to get their perimeters to equal that of C ; none was stretched. But this means that C has a smaller perimeter than any other closed curve enclosing area A . This is what we wanted to show.

Problem 9 (Student page 138) We need to fence in 100,000 square feet using only vertical and horizontal segments of fence. The answer is that we will minimize the perimeter if we use a square fence with sides of length $\sqrt{100,000} = 100\sqrt{10}$.

This habit of using what has been previously done to solve a problem is a very important one!

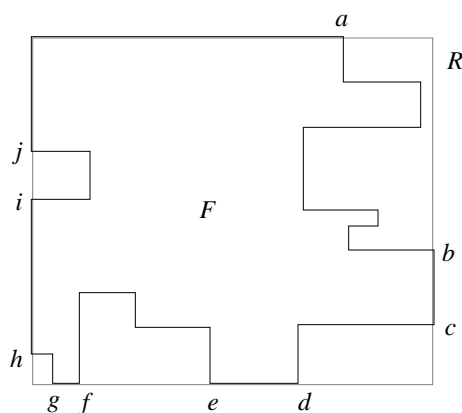
Is it possible to have a fence composed of an infinite number of segments yet enclosing a finite area?

The first step in the solution is to show that the desired fence must be a rectangle. This *seems* pretty clear; if we build the fence with lots of zigzags in it, we will just be using an excessive amount of material and needlessly increasing the perimeter. The idea is that the fence with the smallest perimeter will have a minimal number of zigzags—it will be a rectangle.

In the second step, we once again take advantage of previous work. We already know that, of all rectangles with a fixed area, the square has the smallest perimeter. (Refer back to Problem 24 of Investigation 6.3 for a proof.) So, once we show that the best fence is a rectangle, we can immediately conclude that it is a square.

Let's focus on the first step in the solution, showing that the fence must be a rectangle. This is one of those mathematical facts which seems to just “make sense” but is kind of tricky to prove. Suppose we have any fence which is composed strictly of horizontal and vertical segments, encloses 100,000 square feet, and is not a rectangle. Call this fence F . We want to show that there is some rectangle R which also encloses 100,000 square feet but has smaller perimeter than F . To simplify matters, we shall safely assume that the fence F has a finite perimeter and a finite number of horizontal and vertical segments.

Take R to be the smallest rectangle which completely encloses F (see the picture below). There will be several vertices of F which also lie on R . In fact there will be at least one such point on each of the four sides of R , because if there weren't, then we could find a smaller rectangle containing F . Label these points a , b , c , and so on.



Our plan is as follows:

- Starting at point a , we travel clockwise around the perimeter of R , measuring the length of the path along R from a to b , and then from b to c , and then from c to d , and so on. We then travel clockwise around the fence F , again measuring the lengths of the paths a to b , b to c , c to d , and so on.
- Notice that the paths along R between adjacent points always have lengths less than or equal to the lengths of the paths along F between the same points.
- The above lets us conclude that the perimeter of R is less than or equal to the perimeter of F .
- The rectangle R must have area greater than 100,000, since R contains F . Now shrink R until it contains exactly 100,000 square feet; let R' be the resulting new rectangle. The shrinking decreased the perimeter, so R' is a rectangle containing 100,000 square feet but having a smaller perimeter than R . Therefore, R' has a smaller perimeter than F ; this is exactly what we need to show.

These items let us conclude that the fence with the smallest perimeter enclosing the desired area must be a rectangle; given any other fence, we found a rectangular fence that is better.

The second item on the list above may be the tricky one—seeing that the path between any two points along R will be less than or equal to the path between the same two points along F . If the two points are on the same side of R , then it is clear: the shortest distance between them is along a straight line, and that line contains the side of the rectangle R . If the two points are on adjacent sides of R , then the path between them which lies on R will consist of a horizontal line segment followed by a vertical one, or a vertical line segment followed by a horizontal one. No other path between the points consisting of vertical and horizontal line segments will have shorter length than this path along R ; any such path has to cover the same horizontal and vertical distance, and may likely backtrack and cover more distance and hence be longer.

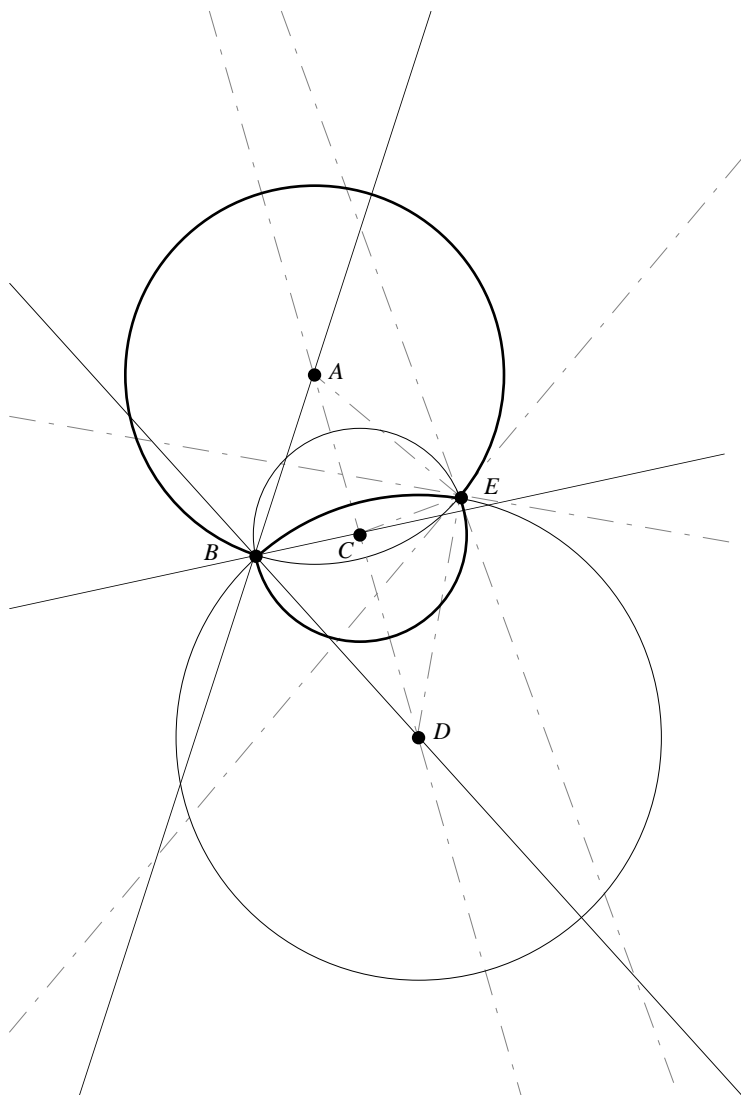
For Discussion (*Student page 139*) These problems are designed to show something of the type of work current mathematicians do, and how this work, though very difficult, is related to the problems in previous investigations. They are not necessarily meant to be solved, but to be discussed and thought about.

Problem 1 (*Student page 140*) Let P be the point where the two arcs meet. Then construct the tangent lines to both the first and the second arcs at P . The angle between these two lines will be the angle between the two arcs.

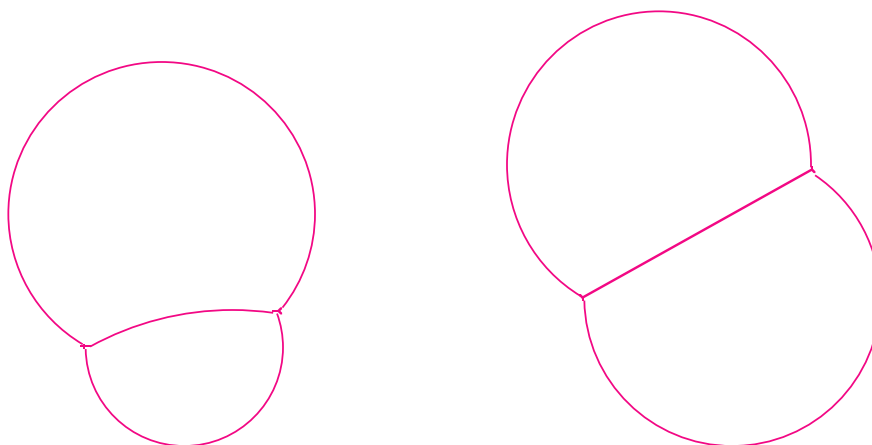
How might one think of this? Up to this point, the only angles discussed have been angles between lines. In order to use what you already know, you have to introduce lines into this problem.

Problem 2 (*Student page 140*) Here is a construction for a double-bubble, made with geometry software:

1. Create three lines that meet at 120° angles. The three lines that started this construction intersect at point B .
2. Create two circles, each with a center on one of the lines, and using B as a radius point. In this sketch, the two circles have centers A and C . They intersect at another point, labeled E in the sketch shown on the next page.
3. Construct a line through the centers of the two circles, and find its intersection with the remaining line through B . That makes point D in the sketch.
4. Use point D as the center of a third circle that passes through points B and E .
5. Create the appropriate arcs on each circle, from point B to point E . The intended arcs are thicker in the sketch on the following page. The arcs will meet at 120° angles to form a “standard double bubble.”

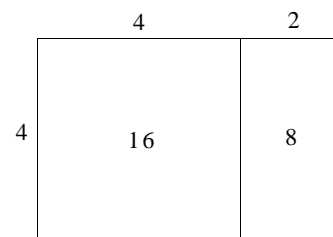
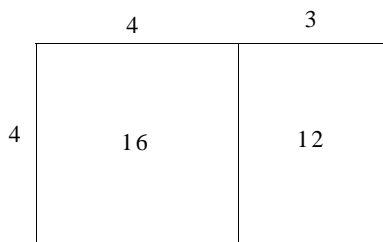


Problem 3 (Student page 142) The shape that minimizes perimeter while enclosing two prescribed areas is the double bubble. It is formed by three arcs, each of a different circle, that form 120° angles with each other. Below are two examples:

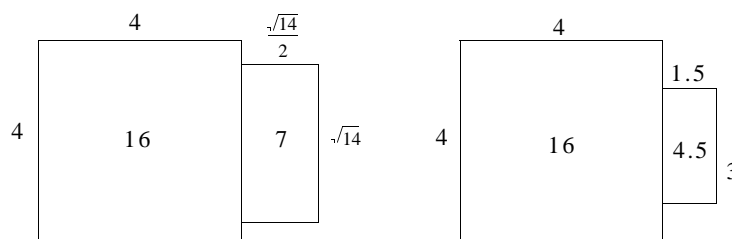


This is one of the questions Scott Greenleaf's group worked on. We might call the answer to this problem something like the "double rectangle."

Problems 4–5 (Student page 142) If the two fixed areas are of equal measure, then the figure will look like two congruent squares attached along one side, so that the perimeter is made up of seven sides. But, if one of the areas is larger than the other, then the overall shape depends upon the ratio of the two areas. If the first area is the largest, then the first area will always take the shape of a square. If the second area is not less than $\frac{1}{2}$ the first area, then the two rectangles will completely share one side.



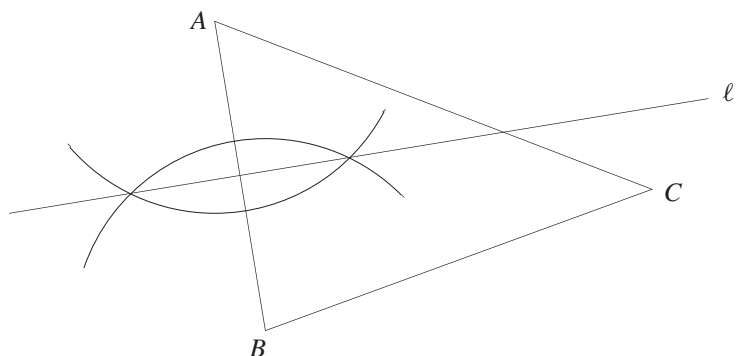
But as soon as the second area is $\leq \frac{1}{2}$ the first area, that second area takes the shape of a rectangle the dimensions of which have the ratio $\frac{1}{2}$. This is the generalized answer to the pen-against-the-wall problems (Problems 6–7 of Investigation 6.3).



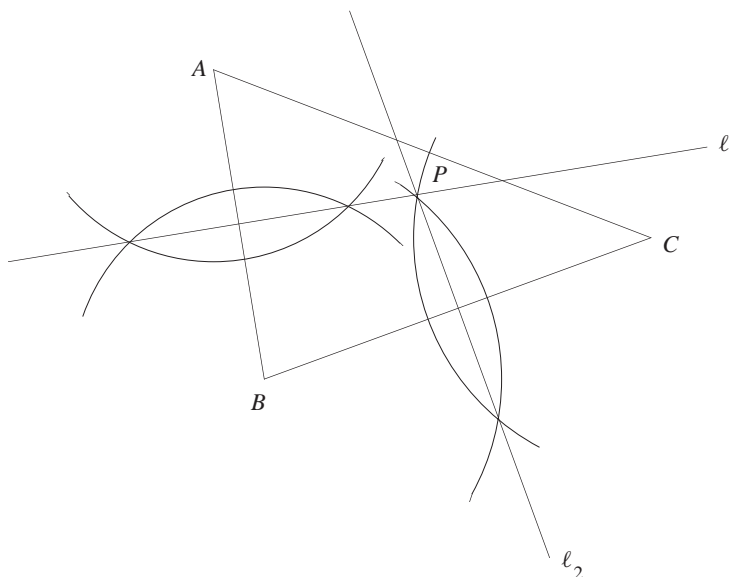
The perimeters of these shapes, which are restricted to horizontal and vertical paths, may be called “taxicab” perimeters. The geometric system which allows measurement of distance only along horizontal and vertical paths may be called “taxicab geometry” because, when traveling by car, you can’t always take the shortest route—you have to drive on the roads, which in a city usually run north to south and east to west!

Problem 6 (Student page 143) Suppose you have a triangle, $\triangle ABC$, and wish to construct a circle passing through its vertices. We know that such a circle will exist, as any three points determine a unique circle passing through them.

First construct the perpendicular bisector of \overline{AB} ; call it ℓ_1 . Recall how to do this: using a compass draw a portion of a circle with center at A , and then draw a portion of a circle with the same radius, with center at B . Draw enough of the circles so that they intersect in two places. Connect these two points to form the perpendicular bisector of \overline{AB} .



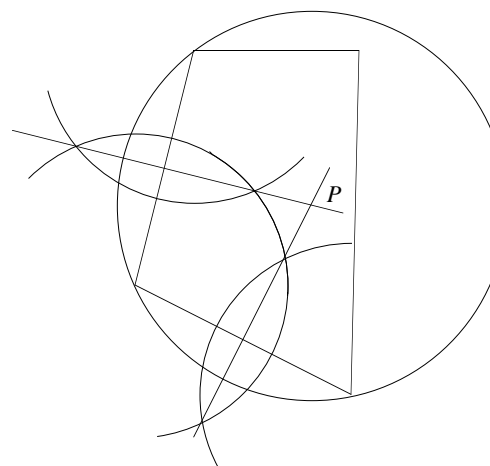
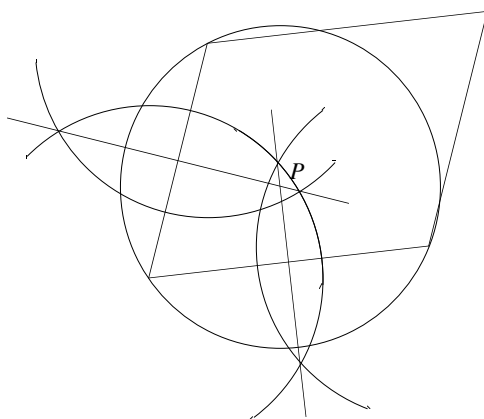
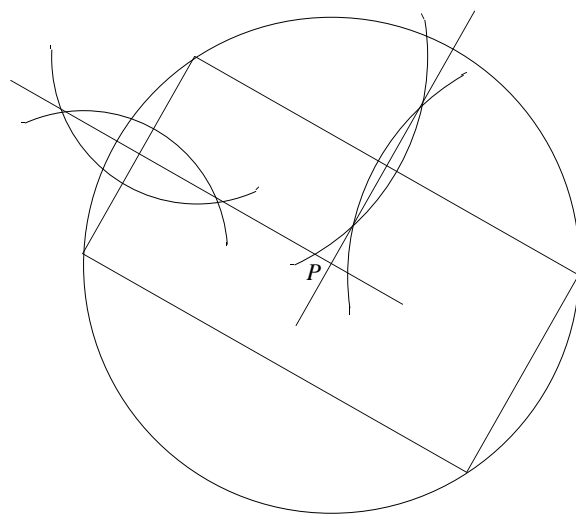
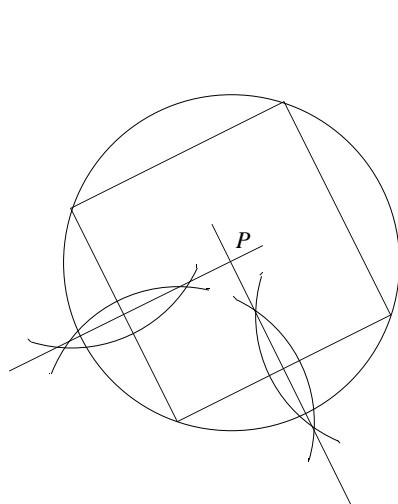
Similarly, form the perpendicular bisector of \overline{BC} and call it ℓ_2 . Label the point of intersection of ℓ_1 and ℓ_2 point P . Now draw the circle centered at P with radius the length of \overline{PA} ; this circle will pass through all three vertices of the triangle. (See Problem 7a for a proof).



The point P above is called the *circumcenter* of $\triangle ABC$. It has the special property of being equidistant from all three vertices. It is a fact that, if ℓ_3 is the perpendicular bisector of \overline{CA} , then ℓ_3 will also intersect ℓ_1 and ℓ_2 at P . The position of the circumcenter can vary greatly—check it out for a smattering of different triangles. It turns out that the circumcenter of an obtuse triangle is outside of the triangle, and the circumcenter of a right triangle is on the hypotenuse (recall that the midpoint of the hypotenuse is equidistant from its vertices).

For the second part of this problem, draw an assortment of quadrilaterals to see which of them allow a circle to pass through all four of the vertices. One way to do this is to first look at only three of the vertices. We know from above how to construct the

unique circle passing through them, and we just need to check if this circle passes through the fourth vertex. If this circle doesn't work, then no other one will, as any circle passing through all four vertices would have to pass through the first three, and there's only one circle which does that. Here are some examples:



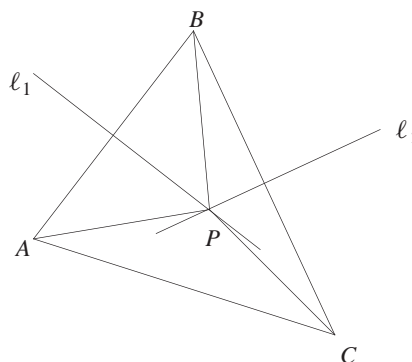
Notice that, for the examples above involving a square and a rectangle, the circle we constructed did in fact pass through all four vertices. Why is this? In both of those polygons, the diagonals are equal and bisect each other. This means that the point P , where the diagonals intersect, is equidistant from all four vertices, so a circle centered

at P will pass through all four. The solution for Problem 7 below tells how to tell if an arbitrary quadrilateral is cyclic.

Problem 7 (Student page 143)

- a. Suppose we have a triangle $\triangle ABC$. We want to show that the triangle is cyclic. Let ℓ_1 and ℓ_2 be the perpendicular bisectors of \overline{AB} and \overline{BC} , respectively, and let P be their point of intersection. Draw line segments \overline{PA} , \overline{PB} , and \overline{PC} .

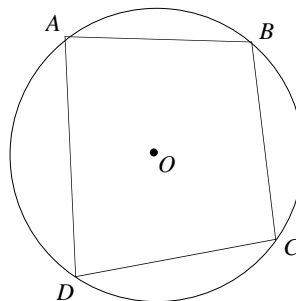
If we reflect \overline{PA} about ℓ_1 , we get \overline{PB} . Since such a reflection preserves distance, we know that $PA = PB$. If we reflect \overline{PB} about ℓ_2 we get \overline{PC} , so $PB = PC$. Therefore, $PA = PB = PC$, meaning P is equidistant from all three vertices. Hence, the circle centered at P with radius PA will pass through all the vertices, so the triangle is cyclic.



- c. Experimenting with different quadrilaterals using some kind of geometry software could be pretty useful here in coming up with a conjecture that a quadrilateral is cyclic if and only if its opposite angles are supplementary. A proof of this follows.

As we've said before, there's a huge difference between being able to work through a proof and being able to produce it yourself. Students reading this proof needn't think we expected more than their conjecture at this point.

Let $ABCD$ be a quadrilateral. First we show that if $ABCD$ is cyclic, then its opposite angles are supplementary. It suffices to just show this for angles A and C , since the same proof will work for angles B and D . Since the quadrilateral is cyclic, we know there is a circle passing through all four vertices. Call the center of the circle O .



Now we use the fact that the measure of an inscribed angle of a circle equals one half the measure of the arc it intercepts. This means that

$$m\angle A = \frac{1}{2}m\widehat{BCD}$$

and

$$m\angle C = \frac{1}{2}m\widehat{BAD}.$$

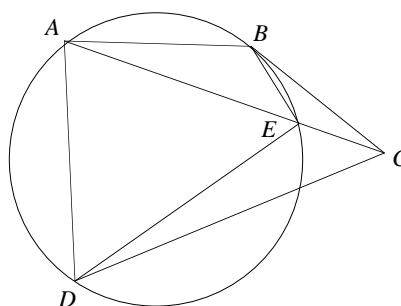
Thus,

$$m\angle A + m\angle C = \frac{1}{2}(m\widehat{BCD} + m\widehat{BAD}) = \frac{1}{2}(360^\circ) = 180^\circ$$

since arcs BCD and BAD together form the entire circle. Thus, angles A and C are supplementary, and a similar proof shows that angles B and D are also supplementary.

Now we are going to show that if the opposite angles are supplementary, then quadrilateral $ABCD$ is cyclic. So, assume that $m\angle DAB + m\angle BCD = 180^\circ$ and $m\angle ABC + m\angle ADC = 180^\circ$. Pick three vertices, say A , B , and D ; draw the unique circle passing through these three points. Three things could happen: vertex C is either on the circle, inside the circle, or outside the circle. In order for $ABCD$ to be cyclic, we need to show that C is on the circle, so we have to eliminate the other two possibilities.

Let's see what happens if C is outside the circle. Draw \overline{AC} and let E be the point of intersection of \overline{AC} with the circle. Draw \overline{EB} and \overline{ED} .



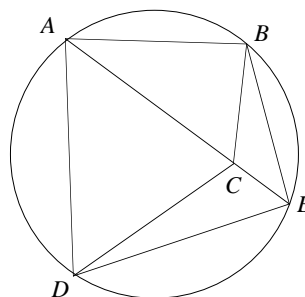
Look at quadrilateral $ABED$; it is cyclic since all vertices lie on the circle. This means, by our proof on the previous page, that its opposite angles are supplementary. So we know that $m\angle DAB + m\angle DEB = 180^\circ$. Remember that by our beginning assumption $ABCD$ has supplementary opposite angles, so $m\angle DAB + m\angle BCD = 180^\circ$. This means that

$$180^\circ = m\angle DAB + m\angle DEB = m\angle DAB + m\angle BCD.$$

This tells us that $m\angle DEB = m\angle BCD$.

But now remember an important fact (see Problems 19–21 of Investigation 6.2). Since C is outside the circle, $\angle BCD$ is smaller than if C were on the circle. This means that $m\angle BCD < m\angle DEB$, contradicting the conclusion above that the two angles have the same measure. Since this is a contradiction, it means that we can't have C outside the circle.

We're almost done. Now let's see what happens if C is inside the circle. Again draw \overline{AC} and continue it until it intersects the circle, calling this point E . Draw \overline{EB} and \overline{ED} .



Again we see that $ABED$ is a cyclic quadrilateral, so by our first proof we know that $m\angle DAB + m\angle DEB = 180^\circ$. Remember from our assumption that the opposite angles of $ABCD$ are supplementary, so $m\angle DAB + m\angle BCD = 180^\circ$. Thus,

$$180^\circ = m\angle DAB + m\angle DEB = m\angle DAB + m\angle BCD.$$

This implies that $m\angle DEB = m\angle BCD$.

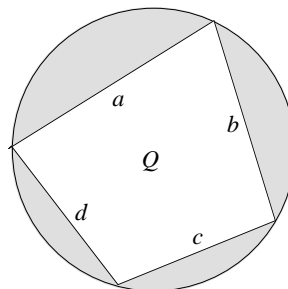
Now we need another important fact, namely that $\angle BCD$ is larger than it would be if C were on the circle. In other words $m\angle BCD > m\angle DEB$, giving a contradiction once again.

So we've seen that we get into trouble if C is either inside or outside of the circle. Therefore, C must be on the circle, which means quadrilateral $ABCD$ is cyclic. This completes the proof.

Problem 8 (Student page 144) If four lengths are fixed, say a , b , c , and d , then the quadrilateral with sides of those lengths containing the most area will be a cyclic quadrilateral. There's an interesting proof of this which nicely makes use of the isoperimetric problem.

One fact we need for the proof is that, given the four values, we can always find a quadrilateral with sides of these four lengths which is cyclic. This seems possible using the main fact we know about cyclic quadrilaterals, which is that the opposite angles have to be supplementary. Imagine we have four sticks of lengths a , b , c , and d . We should be able to find an arrangement of the sticks so that they form a quadrilateral whose angles have this desired property. This quadrilateral will then be cyclic. There's a more rigorous proof that a cyclic quadrilateral with these given sides exists; it uses the Law of Cosines, and is given at the end of this solution for those classes in which students have already studied some trigonometry.

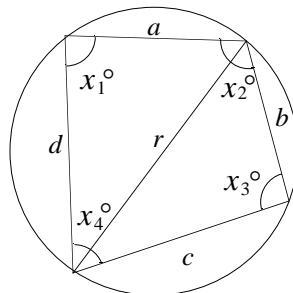
Suppose that lengths a , b , c , and d are given. Let Q be a cyclic quadrilateral with sides of these lengths and let C be the circle in which Q is inscribed (see picture below). We want to show that Q contains more area than any other quadrilateral having sides of these lengths.



Notice that any quadrilateral with sides of length a , b , c , and d can be formed by deforming Q ; imagine that Q has hinges at its vertices, and by opening and closing these hinges we can get any other such quadrilateral. As the hinges open and close, the circle C will be deformed to other closed curves in which the new quadrilaterals are inscribed. The area outside the quadrilaterals and inside the curves (the shaded area in the picture) will not change, however, nor will the perimeter of the curves. So by manipulating the hinges we get all quadrilaterals with sides of lengths a , b , c , and d , and we get different closed curves with the same perimeter as C . We will have found the quadrilateral with maximal area precisely when we have the closed curve with maximal area (because the shaded area is fixed). Now we use the isoperimetric problem. The area inside the curve will be largest when the curve is the circle C . This will happen when the quadrilateral is the cyclic quadrilateral Q , which is what we wanted.

Here is a trigonometric proof that given a , b , c , and d , we can always find a cyclic quadrilateral with sides of these lengths. Form a quadrilateral with sides these lengths, and label the angles of the quadrilateral x_1 , x_2 , x_3 , and x_4 (see measures in the picture on the next page).

The quadrilateral will be cyclic when $x_3 = 180 - x_1$ and $x_4 = 180 - x_2$. See the solution for Problem 7c. If we can solve for these angles, then we just arrange the sides of the quadrilateral so as to obtain these four interior angles; then we will have a cyclic quadrilateral.



Let r be the length of the diagonal shown in the figure. Looking at the triangle formed with sides of length a , d , and r , and using the Law of Cosines we see that

$$(1) \quad r^2 = a^2 + d^2 - 2ad \cos x_1.$$

Looking at the triangle with sides of length c , b , and r , and again using the Law of Cosines, we see that

$$\begin{aligned} r^2 &= b^2 + c^2 - 2bc \cos x_3 \\ r^2 &= b^2 + c^2 - 2bc \cos (180 - x_1). \end{aligned}$$

Apply the trigonometric identity $\cos (180 - x) = -\cos x$, we obtain

$$(2) \quad r^2 = b^2 + c^2 - 2bc(-\cos x_1) = b^2 + c^2 + 2bc \cos x_1.$$

Equating the expressions for r^2 from equations (1) and (2), we have

$$r^2 = a^2 + d^2 - 2ad \cos x_1 = b^2 + c^2 + 2bc \cos x_1.$$

So

$$\begin{aligned} (2bc + 2ad) \cos x_1 &= a^2 + d^2 - b^2 - c^2 \\ \cos x_1 &= \frac{(a^2 + d^2) - (b^2 + c^2)}{2(bc + ad)}. \end{aligned}$$

This equation lets us solve for x_1 , and thus for x_3 .

Letting s be the length of the other diagonal and using an argument similar to that above, we can solve for x_2 and x_4 . As we said earlier, these angles completely determine a cyclic quadrilateral with lengths a , b , c , and d .